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A MATHEMATICAL MODEL FOR TURBULENT FLOWS INVOLVING SUPERSONIC, SUBSONIC AND RECIRCULATING REGIONS

by

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In connection with the development of a dual chamber rocket, the need arose for a mathematical model capable of simulating the flow field involved. The flow is turbulent and includes supersonic, subsonic and recirculating regions. Such a model is fully described in this report.

Turbulence effects are accounted for by an eddy viscosity hypothesis, and by suitable coefficients of mass, energy and entropy transport. It was found that these turbulence effects radically change the elliptical/hyperbolic

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This report explains how the field may be subdivided into finite cells and the solution marched downstream cell by cell.

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SUMMARY

In connection with the development of a dual chamber rocket, the need arose for a mathematical model capable of simulating the flow field involved. The flow is turbulent and includes supersonic, subsonic and recirculating regions. Such a model is fully described in this report.

Turbulence effects are accounted for by an eddy viscosity hypothesis, and by suitable coefficients of mass, energy and entropy transport. It was found that these turbulence effects radically change the elliptical/hyperbolic characteristics of the equations as compared with the classical case of nonturbulent compressible flow. The equations of momentum, continuity and energy for turbulent flow are shown to be elliptical for both supersonic and subsonic regions. When the second law of thermodynamics is added, the equations assume a parabolic character.

This report explains how the field may be subdivided into finite cells and the solution marched downstream cell by cell.

1. Purpose and Scope of Report

This report has two principal purposes. The first is simply to summarize the progress that has been made during the fiscal year ending 30 Sept 1978 on the theoretical aspects pertaining to the development of a dual chamber rocket. The associated experimental program is summarized separately by Netzer [62].

The initial objective of the theoretical program was to develop a mathematical model adequate for calculating and predicting the flow and performance characteristics of this type of device. This required a number of initially baffling paradoxes to be resolved, but the desired objective has finally been achieved.

The second principal aim of this report is to summarize the theoretical flow analysis and mathematical model that has finally been developed and to explain in considerable detail the rationale involved.

2. Current Status of the Theoretical Investigation

The initial phase of the theoretical program as reported herein has been successfully completed. It has resulted in a formulation of the problem that may be fairly regarded as a novel contribution to the state of the art of computational fluid mechanics. Moreover, it has not disclosed any decisive barrier to further progress.

Consequently, the next step should be to translate the present mathematical model into a functioning computer code. When this has been accomplished, a systematic program of detailed calculations and comparisons with experiment can be undertaken to explore and delineate the overall performance potential of the dual chamber rocket.

The nature of our mathematical model suggests that the final computer code will entail calculations which, while massive, nevertheless lie within the present state of the art.

In the preliminary phase of the present study, the author reviewed several dozen papers in the recent technical literature and several classical texts to ascertain whether any of the currently existing methods is readily adoptable to the present problem. In this connection, see the references and bibliography listed in section 23. While this review provided much useful background information it failed to disclose any ready made method to do the present job. Hence the author was obliged to tackle the problem from first principles. The effort was successful and the resulting mathematical model is fully described in this report. The principal equations are summarized in section 18. Because of these circumstances, the references cited in the text as well as the additional items listed in section 23, while interesting and helpful, are not essential for understanding the present text.

3. The Dual Chamber Rocket

This report outlines the progress that has been made and summarizes the mathematical model that has been developed for the analysis of a type of turbulent supersonic flow which contains regions of recirculation. While the potential field of applicability of this model is quite broad, the present study arose specifically in connection with efforts to develop an effective small dual chamber solid propellent rocket of a type suitable as an air-to-air weapon. Hence some information about the basic concept of the small dual chamber rocket is appropriate at this point to provide orientation concerning the initial intended application of the present theoretical analysis.

A comparison of the small dual chamber rocket with a large multi-stage rocket is instructive here. It is well known that effective performance of the large multi-stage rocket demands that each stage be jettisoned as soon as its fuel is spent. Ideally, such jettisoning would be desirable for the small rocket as well, but in practice, the incidental penalties in weight and complexity required to accomplish this outweight the basic performance advantages that would be gained from such jettisoning. The problem therefore arises of designing the small two stage rocket to operate effectively without jettisoning the first or booster stage when it is spent. Consequently, the second or sustain stage must fire and discharge through the empty booster casing. The general question therefore arises as to the range of parameters over which such a mode of operation can be made reasonably effective.

One design compromise has been suggested that significantly affects the situation. This is to jettison not the whole booster stage, but only the aft end of the booster which holds the aft nozzle. It turns out that the mechanical complications and penalties required to accomplish this are much less than those required to jettison the entire stage. If this option be elected, the sustain stage discharges through a simple cylindrical tube, open at the aft end. Under these circumstances there is far less interference with the effective performance of the front nozzle. In particular, the permissible length of the booster stage is not as severely restricted in this case. Moreover, the flow through the open tube, while still complex, is substantially simpler than that through an aft nozzle. The reason for this is that, assuming sufficiently low ambient pressure, there would be no complex structure of imbedded shocks in the open tube, owing to the absence of downstream choking. Consequently, our present theoretical analysis emphasizes the open tube case, at least for the time being.

The general nature of the various possible flow regimes in the dual chamber rocket is shown schematically in Figs. 3.1 and 3.2. Fig. 3.1 deals with the configurations in which the aft nozzle remains in place while Fig. 3.2 deals with the configurations in which the aft nozzle has been jettisoned. Notice that in all cases the flow field consists of an expanding, axisymmetric, supersonic inner jet plus an outer annulus of recirculating flow. Velocities over much of the recirculating region are presumably subsonic.

The expanding inner jet may or may not contact the walls of the aft nozzle or of the aft tube before exiting. If the jet does not contact the walls, the recirculating region remains in direct communication with the ambient air and the aft chamber is said to be ventilated. In this case the general pressure level in the recirculating region is governed primarily by the ambient pressure.

If the expanding jet contacts the tube or nozzle wall, it seals off the recirculating region from contact with the ambient air and the aft chamber is then said to be unventilated. The general pressure level in the recirculation region is now governed by the complex mechanisms of turbulent transfer of momentum, mass, energy and entropy between the recirculating fluid and the main jet.

The analysis of such recirculating flows involves special difficulties but is still possible. See, for example, references [26] through [31].

Given specified values of all of the other significant parameters, there exists some corresponding critical length L_{cr} of the aft chamber at which the inner jet just makes effective contact with the outer walls and thereby just seals off the recirculation zone from the ambient air. Unfortunately, it has not been possible up to now to determine the value of L_{cr} by theoretical calculation, but only by experiment.

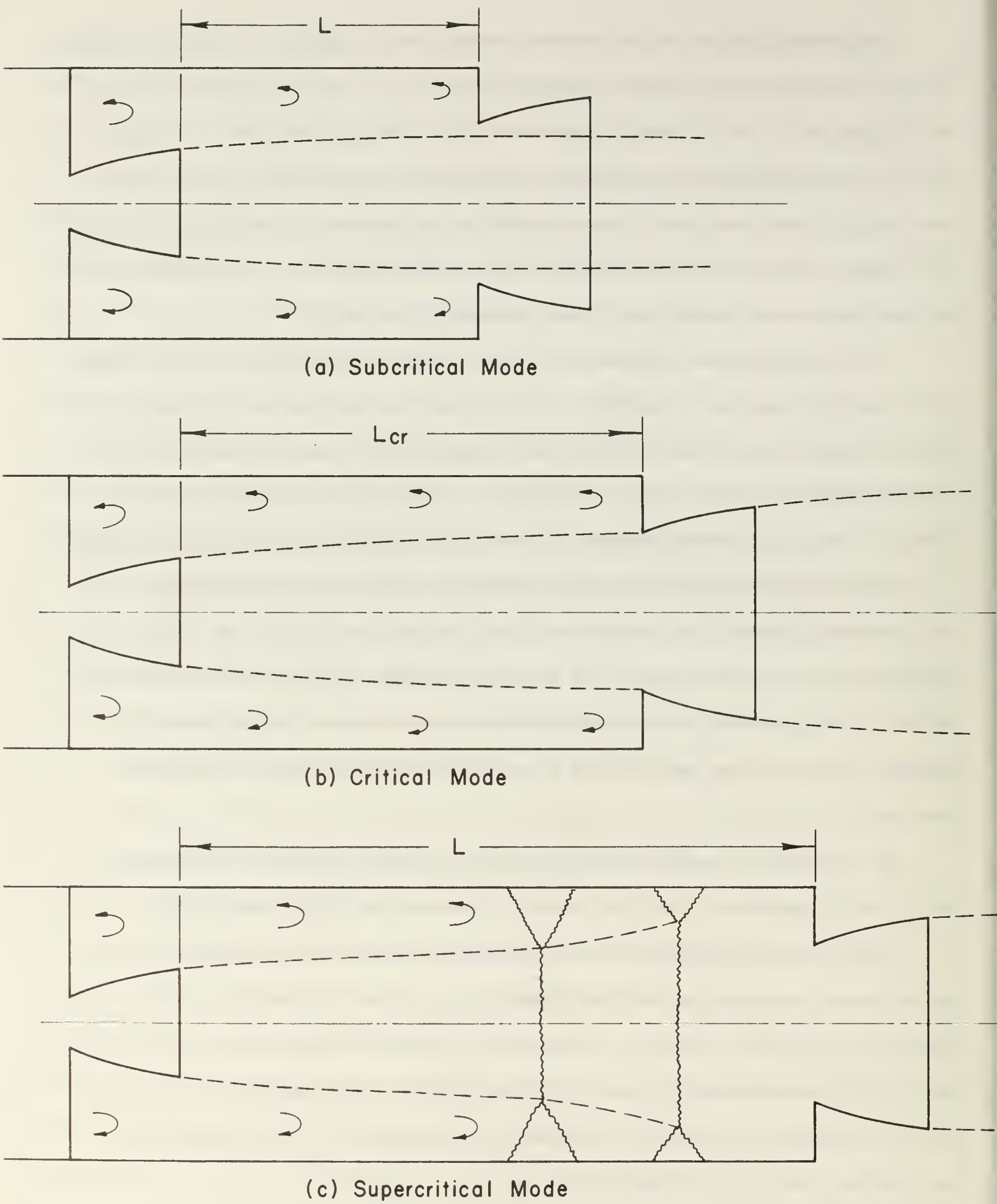


FIG. 3.1 OPERATING MODES WITH AFT NOZZLE IN PLACE

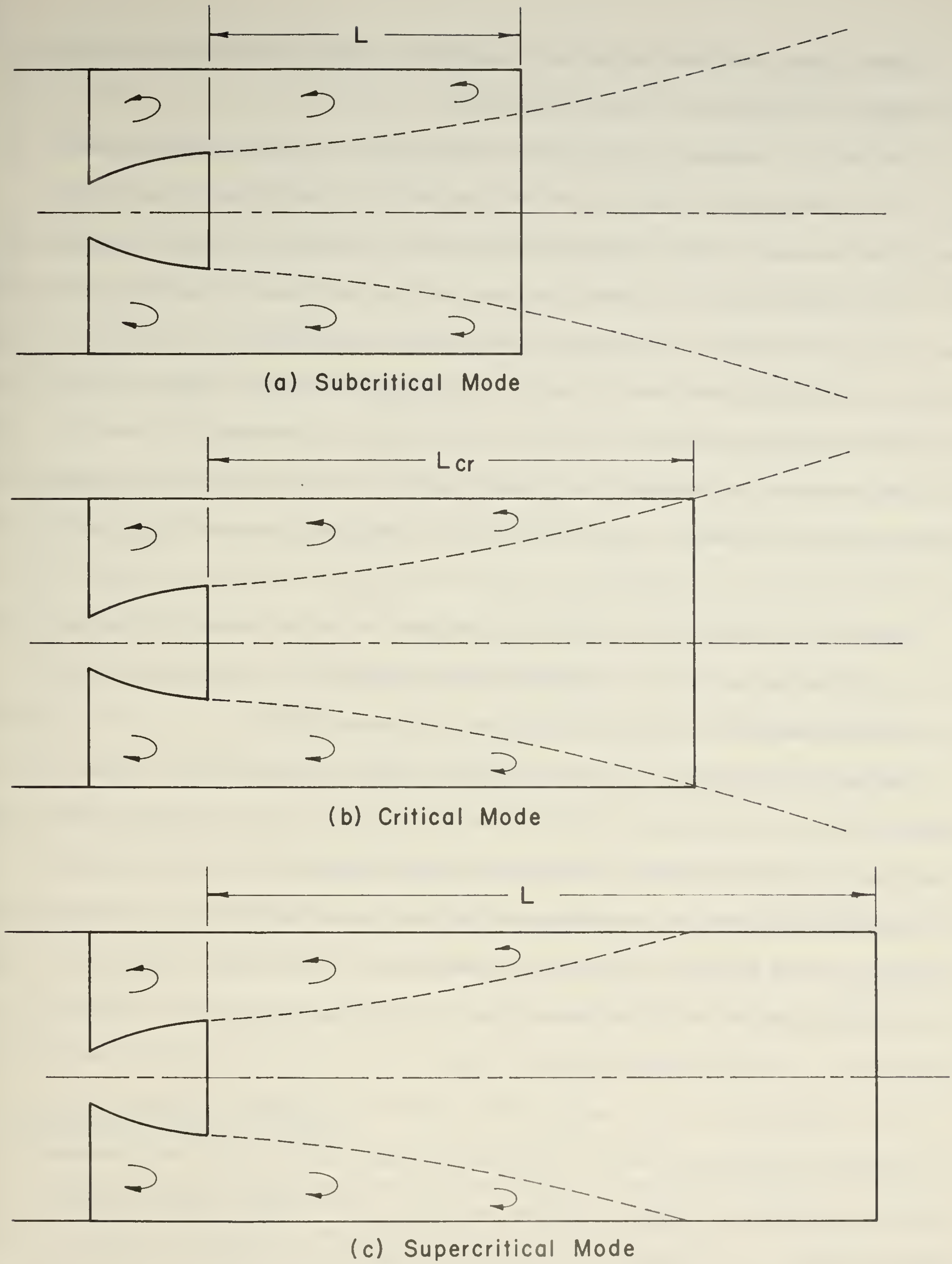


FIG. 3.2 OPERATING MODES WITH AFT NOZZLE JETTISONED

Typical test data bearing on various aspects of jet performance are given, for example, in references [56] through [67].

Such sketchy empirical data as are available on this point suggest that L_{cr} will usually be far smaller than the values of L which are of practical importance because of other design considerations. Hence our primary interest will be in operation in the super-critical modes as indicated schematically in Figs. 3.1(c) and 3.2(c). Of course, these are the most difficult cases to analyze, especially that shown in Fig. 3.1(c) which involves a rather complex structure of imbedded shocks. In fact, just because of these shocks, this case lies outside the scope of the present report. In principle, the present analysis applies, however, to any of the other five cases shown in Figs. 3.1 and 3.2.

Extension of the present methods to include the case of imbedded shocks, Fig. 3.1(c), while difficult, might eventually prove to be practicable. On this question see, for example, the analysis of Murman [38].

The sub-critical modes, Figs. 3.1(a) and 3.2(a), represent the easiest problem because in these cases the jet can be expected to approximate to the classical case of a free jet. Of course, there is considerable experimental and analytical information available on the free jet. Nevertheless, even in this case, if the free jet happens to be supersonic, many significant details of the flow field are still far from clear. On this point see, for example, Abramovich [1] and Kovasznay [35].

It is tempting to try to obtain some limited preliminary information on the performance potential and on the inherent limitations of the dual chamber rocket by considering certain limiting cases on a drastically simplified and idealized basis. For example, one might treat the flow as essentially one

dimensional and inviscid. Also, the rather complicated shock structure shown in Fig. 3.1(c) might be treated as a simple normal shock. A comparison of the cases shown in Figs. 3.1(c) and 3.2(c) could then be carried out.

In all such over-simplified analyses, however, the net forces exerted on the walls by the recirculating fluid cannot be deduced from the fundamental equations, and must be estimated on the basis of some more or less plausible but uncertain ad hoc assumptions. Moreover, such idealized analyses give no reliable information on the crucial matter of the rate of spreading of the jet. This rate is fundamentally determined by the complicated mechanics of turbulent mixing and cannot be adequately analyzed on any basis that ignores the turbulence. Moreover, spreading rate appears to be a sensitive function of the boundary conditions so that any attempt to estimate this parameter for the confined jet on the basis of experimental results for the free jet is apt to be seriously in error. Because of limitations such as these, we do not digress in the present report to explore or present grossly over-simplified calculations of this kind but instead proceed directly to a more comprehensive type of analysis.

Needless to say any research effort of this kind should involve coordinated experimental and analytical aspects. This report deals only with the development of a mathematical model, which is the specific task that has been undertaken by the author. The related experimental work is under the direction of, and is reported separately by Prof. D. W. Netzer [62].

4. Turbulent Transport Effects

Among the various important matters which the mathematical model must be able to analyze and predict are the spatial rate of spreading of the jet and the pressure distribution within the recirculating region of the flow.

Physically, these matters are largely governed by the complex mechanisms of turbulent mixing. Hence to be suitable for its purpose, our mathematical model must account adequately for the turbulent transport of key quantities, specifically of mass, energy, entropy and momentum. Naturally, this requirement introduces unavoidable complications and supplementary questions of various kinds into the analysis. See, for example, references [1] through [25].

To help place the turbulence problem in proper perspective, it is useful to consider briefly first the special case of incompressible turbulent flow. For incompressible flow, various thermodynamic relations either drop out of the analysis entirely or simplify drastically so that attention is more easily focussed on the phenomena characteristic of the turbulence itself.

In principle, the solution of any unsteady incompressible flow is completely determined by the continuity equation, by the Navier Stokes equations and by the appropriate boundary conditions. This applies also to the type of unsteady motion which characterizes turbulence. Unfortunately, however, two circumstances conspire to thwart all efforts to solve these equations for the case of turbulent flow, despite their theoretical claim to sufficiency. Firstly, the basic equations are nonlinear. Secondly, the detailed turbulent motion encompasses a very large range of length scales with respect to each of the three spatial axes, and a very large range of time scales as well. Thus an astronomical number of degrees of freedom is required just to specify the state of the flow field at a given instant of time. An even more prodigous number is required to trace out the evolution of the field over time. Moreover, such a detailed solution, even if it were possible, would provide far, far more data than is needed or usable; assuming a steady mean flow, what is really required is merely the statistically average properties of the solution at each point in the field.

Inasmuch as it is impossible to obtain the detailed solution in order to average it, an alternative is to average the basic equations themselves, and to attempt to solve these. However, because of the nature of the nonlinearities in the present system of equations, the process of averaging always introduces additional unknowns so that we inevitably end up with more unknowns than equations. This is the well known closure problem of turbulence theory. Consequently, in order to define a determinate solution there is an unavoidable necessity to introduce some auxiliary postulates which cannot be shown to follow from the original governing equations themselves. The adequacy of any such auxiliary postulates can then be demonstrated only partially and indirectly by comparison of theory with experiment.

Let us now drop the above restriction to incompressible flow and revert to the fully general case of flow which is both compressible and turbulent. It is still necessary to average the governing equations and this process still gives rise to additional unknowns. If the mean flow be steady this average, which is symbolized in the usual way by an overbar, may be thought of as a simple time average and this simple interpretation is adequate for the present discussion. If the mean flow be unsteady, a more sophisticated ensemble average becomes necessary, but this further generalization need not concern us here.

Let us now examine in more detail the nature of the additional unknowns that arise from the averaging process. Consider first the equation of continuity. Here the density ρ is the fluid property of primary significance. To distinguish between average and fluctuating values, we utilize the following notation.

ρ = the average value of the density at a particular point

ρ' = the instantaneous deviation of the density from its average value at the point

ρ'' = the instantaneous fluctuating value of the density at the point in question

According to these definitions

$$\rho'' = \rho + \rho' \tag{4.1}$$

Using an overbar to denote a time or ensemble average as appropriate, we may also write

$$\bar{\rho}'' = \bar{\rho} = \rho \tag{4.2}$$

and

$$\bar{\rho}' = 0 \tag{4.3}$$

In the later detailed analysis it is natural and convenient to use cylindrical coordinates, but in the preliminary discussion of this section it is actually simpler and clearer to employ cartesian axes and cartesian tensor notation. Thus symbols x_1, x_2, x_3 denote the axes and u_1', u_2', u_3' denote the corresponding components of the velocity fluctuation. We may also denote these quantities simply as x_i and u_i' where $i = 1,2,3$.

Using the foregoing notation, we find that the continuity equations upon being suitably averaged, contains terms of the form $\overline{u_i'\rho'}$. This is the type of additional unknown which results from the averaging process. This quantity may be said to represent the net turbulent transport of fluid mass in the direction of axis x_i.

It is customary to assume that such turbulent transport can be adequately described by a relation of the form

$$\overline{u_i'\rho'} = -\varepsilon_\rho \left(\frac{\partial \rho}{\partial x_i}\right) \tag{4.4}$$

$$i = 1,2,3$$

where ε_ρ is termed the turbulent transport coefficient. Notice that ε_ρ is treated as a true scalar, that is, as a quantity whose magnitude is independent of the orientation of axis x_i. On the other hand, the value of ε_ρ may in general vary from point to point in the flow.

However, it is known from experiment that for an axi-symmetric free jet, all turbulent transport coefficients, like ε_ρ in the present discussion, remain approximately constant over most of the flow field. Presumably this simple assumption applies also to the confined jet, at least as a first approximation. Experimental support for this assumption is given by Abramovich [1] and Schlichting [2].

Similar consideration also apply in connection with the first law of thermodynamics. The significant fluid property in this case turns out to be total energy per unit volume, denoted by symbol Q. This in turn involves the mean kinetic energy of turbulence at the point in question, denoted by symbol ρE. The following definitions apply

$$\rho E = \frac{1}{2} (\overline{\rho'' u''_k u''_k} - \rho\, u_k u_k) \tag{4.5}$$

$$Q = \rho \left[e + \left(\frac{u_1^{\,2} + u_2^{\,2} + u_3^{\,2}}{2} \right) + E \right] \tag{4.6}$$

where e denotes the ordinary static internal energy per unit mass. Notice that the summation convention applies to the repeated index k in Eq. (4.5).

When the averaged energy equation is examined, it is found to contain turbulent transport terms of the form $\overline{u'_i Q'}$. Hence by analogy with Eq. (4.4) we write

$$\overline{u'_i Q'} = - \varepsilon_Q \left(\frac{\partial Q}{\partial x_i}\right) \tag{4.7}$$

$$i = 1,2,3,$$

Proceeding next to the second law of thermodynamics, we define the entropy per unit volume S in terms of the entropy per unit mass s as follows

$$S = \rho s \tag{4.8}$$

When the corresponding averaged equation is examined, it is found to contain turbulent transport terms of the form $\overline{u_i' S'}$. We again express these in the form

$$\overline{u_i' \, S'} = - \varepsilon_S \left(\frac{\partial S}{\partial x_i}\right) \tag{4.9}$$

$$i = 1,2,3,$$

It also simplifies matters to assume that at any given point in the field

$$\varepsilon_\rho = \varepsilon_Q = \varepsilon_S \tag{4.10}$$

This is plausible since all three of these coefficients reflect the same basic physical process of turbulent mixing. Moreover, all three of the fluid properties ρ , Q and S which are transported by the turbulence are scalar quantities, and all are expressed on a per unit volume basis.

On the other hand the process of averaging the momentum equations for compressible turbulent flow gives rise to a set of momentum transport quantities which in the present notation turn out to have the following formidable looking algebraic structure, namely,

$$\begin{aligned}
\tau_{ij} = & \left\{- \overline{\rho' u_i' u_j'} + \frac{1}{3} \delta_{ij} \overline{\rho' u_k' u_k'}\right\} \\
& + \rho \left\{- \overline{u_i' u_j'} + \frac{1}{3} \delta_{ij} \overline{u_k' u_k'}\right\} \\
& + \left\{- u_i \overline{\rho' u_j'} - u_j \overline{\rho' u_i'} + \frac{2}{3} \delta_{ij} u_k \overline{\rho' u_k'}\right\}
\end{aligned} \tag{4.11}$$

$$i, j = 1,2,3,$$

In Eq. (4.11) symbol δ_{ij} denotes the Kronecker delta which is defined as follows

$$\delta_{ij} = +1 \quad \text{if} \quad i = j \qquad\qquad (4.12)$$
$$= 0 \quad \text{if} \quad i \neq j$$

Also, the usual summation convention of tensor analysis applies to the repeated index k in Eq. (4.11). For example

$$\overline{u'_k u'_k} = \overline{u'^2_1} + \overline{u'^2_2} + \overline{u'^2_3} \qquad (4.13)$$

Inspection of Eqs. (4.11) further reveals that all terms have the dimensions of stress. In fact the quantities τ_{ij} defined by Eq. (4.11) are the familiar Reynolds stresses. These are additional unknowns created by the averaging process. They are seen to constitute a symmetrical and purely deviatoric tensor of second order.

In order to effect closure of the overall system of equations, we must postulate a suitable governing relation for these quantities, most conveniently one which, like Eq. (4.4), (4.7) or (4.9), relates the net turbulent transport to appropriate local gradients of the mean flow field through one or more suitable transport coefficients.

Notice that the last pair of brackets in Eq. (4.11) offers no difficulty in this regard because the quantities under the overbars conform to the format already established in Eq. (4.4). This is not true, however for the quantities under the overbars within the first two pairs of brackets. The distinction is that in the third pair of brackets we are dealing with the transport of a scalar which is a vector, while in the first two pairs we are dealing with the transport of a vector which is a second order tensor.

The customary way of handling this complication is to postulate a relation between the Reynolds stresses and the strain rates of the mean flow which is analogous to the relation which is known to apply between the viscous stresses and the strain rates. In the present application this analogy must be generalized slightly to allow for the peculiarities introduced by compressibility. This can be done by casting the assumed relation into the following form

$$\begin{aligned}\tau_{ij} = \varepsilon' & \left\{\left[\frac{\partial}{\partial x_i}(\rho\, u_j) + \frac{\partial}{\partial x_j}(\rho\, u_i)\right] - \frac{2}{3}\,\delta_{ij}\frac{\partial}{\partial x_k}(\rho\, u_k)\right\} \\ & + \varepsilon''\rho\left\{\left[\left(\frac{\partial u_j}{\partial x_i}\right) + \left(\frac{\partial u_i}{\partial x_j}\right)\right] - \frac{2}{3}\,\delta_{ij}\left(\frac{\partial u_k}{\partial x_k}\right)\right\} \\ & + \varepsilon_\rho\left\{\left[u_i\left(\frac{\partial \rho}{\partial x_j}\right) + u_j\left(\frac{\partial \rho}{\partial x_i}\right)\right] - \frac{2}{3}\,\delta_{ij}u_k\left(\frac{\partial \rho}{\partial u_k}\right)\right\} \qquad (4.14)\end{aligned}$$

where ε' and ε'' are appropriate momentum transport coefficients and ε_ρ is the scalar transport coefficient defined earlier in connection with Eq. (4.4).

Notice that of the three sets of terms on the right side of Eq. (4.14), the second set is exactly of the same form as is known to apply to the viscous stresses. However, the corresponding eddy viscosity, here denoted as $\varepsilon''\rho$, is many times greater than the ordinary molecular viscosity.

The first set of terms on the right side of Eq. (4.14) is derived from the second set merely by replacing each velocity component u_i by the corresponding mass velocity component ρu_i. The reason for this shift can be discerned by studying the corresponding sets of terms in Eq. (4.11).

The use of three distinct transport coefficients in Eq. (4.14), namely, ε', ε'' and ε_ρ, is unduly elaborate. There are grounds for assuming that ε' and ε'' can be treated as equal. Moreover, further information can be

gained from the special case of incompressible flow. In this event the first two sets of terms on the right side of Eq. (4.14) reduce to identical form and the third set vanishes. This strongly suggests that we may place

$$\varepsilon' = \varepsilon'' = \frac{\varepsilon}{2} \tag{4.15}$$

where ε will now be termed the kinematic eddy viscosity.

The relation between the three coefficients $\varepsilon_\rho = \varepsilon_Q = \varepsilon_S$, all of which pertain to the transport of a scalar property, and the momentum coefficient ε, which pertains to the transport of a vector property, cannot be predicted from the present theory. However, physical considerations and test data suggest that, although both may vary from point to point, their ratio may be treated as approximately constant. Hence we hereafter write

$$\varepsilon_\rho = \varepsilon_Q = \varepsilon_S = \kappa\varepsilon \tag{4.16}$$

where κ is taken as constant over the flow field. The constant κ may be recognized as the reciprocal of the turbulent Prandtl number. Parameter κ usually lies somewhere between 1.4 and 2. We shall take $\kappa = 2$ in initial trial calculations.

We can now substitute Eqs. (4.15) and (4.16) into Eq. (4.14), then expand the latter and regroup terms. In this way we finally obtain the result

$$\begin{aligned}\tau_{ij} = \varepsilon\rho \Bigg\{ &\left[\left(\frac{\partial u_j}{\partial x_i}\right) + \left(\frac{\partial u_i}{\partial x_j}\right) - \frac{2}{3}\,\delta_{ij}\left(\frac{\partial u_k}{\partial x_k}\right)\right. \\ &+ \left(\frac{1}{2} + \kappa\right)\left[u_i\left(\frac{\partial \rho}{\partial x_j}\right) + u_j\left(\frac{\partial \rho}{\partial x_i}\right) - \frac{2}{3}\,\delta_{ij} u_k\left(\frac{\partial \rho}{\partial u_k}\right)\right]\Bigg\}\end{aligned} \tag{4.17}$$

$i, j = 1,2,3$

Likewise Eqs. (4.4), (4.7) and (4.9) can now be rewritten in the slightly simplified form

$$\overline{u_i'\rho'} = -\kappa\varepsilon\left(\frac{\partial\rho}{\partial x_i}\right)$$

$$\overline{u_i'Q'} = -\kappa\varepsilon\left(\frac{\partial Q}{\partial x_i}\right) \tag{4.18}$$

$$\overline{u_i'S'} = -\kappa\varepsilon\left(\frac{\partial S}{\partial x_i}\right)$$

Eqs. (4.17) and (4.18) summarize in a unified manner the general format adopted for the closure hypothesis. In general κ is taken as an empirical constant, while ε may vary from point to point in the field.

This turbulence model is not complete until the relations are stipulated which fix the actual distribution of ε over the field. As a rule, ε may be regarded as some function of the turbulent kinetic energy E and of a second turbulence parameter, usually some suitable characteristic length scale λ or rate of dissipation of turbulent energy D.

We shall not consider these more involved aspects of the turbulence model in this report for two reasons. Firstly, there are more fundamental and urgent questions that have higher priority. Secondly, we have the good luck to be dealing with a case for which the simple assumption of a constant eddy viscosity over the field appears to provide an adequate basis for the initial computational trials. Support for this assumption may be found in the texts by Abramovich [1] and by Schlicting [2].

It is perhaps worth pointing out that any and all turbulence closure models are inherently inexact by their very nature. This is true of Eqs. (4.17) and (4.18) as well. Nevertheless, if the distribution of ε be specified with

sufficient care, an eddy viscosity model of the present type is capable of representing a wide range of flows with an overall accuracy sufficient for most engineering purposes. This general conclusion is supported by the experience of a growing number of investigators over the past two decades. See, for example, references [10] through [25], and Reynolds [43, 44].

5. Choice of Fundamental Variables

Some problems in fluid mechanics are formulated in terms of a velocity potential ϕ, others in terms of a stream function ψ, still others in terms of the velocity components u and v and the pressure p and so on. What is an appropriate choice of independent variables for the present problem?

Of course the use of the velocity potential ϕ is restricted to irrotational motions which rules out this particular method in the present case.

Inasmuch as the present problem is restricted to mean motions which are axisymmetric, the use of a stream function ψ is a possibility. Of course, cylindrical coordinates are appropriate here. This method has the advantage of expressing the two non-zero components u and v of the mean velocity in terms of the single variable ψ. It is also customary in this case to employ the vorticity transport equation, rather than the two momentum transport equations themselves, in order to eliminate the pressure p from the fundamental equations. These convenient features are of course somewhat counterbalanced by certain corresponding disadvantages. One of these is the difficulty of extracting accurate pressure distributions from the calculated stream functions.

The possibility of analyzing the present problem in terms of a stream function ψ was studied with some care, but was ultimately rejected. It was found that while the steady flow equations could be formulated in these terms,

the problem of finding a relaxation procedure that would ensure proper convergence of the resulting equations to a stable solution proved troublesome.

To get around this difficulty it was decided to reformulate the equations in the form that applies when the mean motion is unsteady. In this case the solution defined by the equations automatically approaches the proper steady state as the calculation are allowed to advance in time. While this convergence is perhaps not as rapid as we should like, it does finally occur. Moreover, computational strategies can be devised to speed up the convergence rate if necessary.

On the other hand, in relation to this flow field which is now both compressible and unsteady, the whole concept of defining the velocity components u and v in terms of a stream function ψ breaks down. A stream function simply cannot be defined for an unsteady compressible flow. Hence we have to abandon the use of the stream function ψ under these circumstances and work directly in terms of the velocity components u and v as independent variables.

Recall also that when the stream function ψ is employed, the vorticity transport equation is used to eliminate pressure p from the analysis, and to give a governing equation expressed primarily in terms of ψ. Naturally, when ψ is dropped, this step becomes more or less pointless; it becomes simpler to retain variable p explicitly in the analysis.

In most problems of supersonic flow, the fluid may be treated as inviscid outside the boundary layer. Consequently, the flow is usually irrotational and the entropy uniform over the field. With such a restriction on entropy, the single additional thermodynamic property p suffices to fix the thermodynamic properties of the fluid at every point in the field. Thus the fundamental variables are u, v and p in this case.

The presence of strong turbulence changes these conditions. It introduces a viscous-like action over the entire field and causes the entropy to vary from point to point. Thus, p itself no longer suffices to fix the thermodynamic state at an arbitrary point. Two independent thermodynamic properties are now needed. These could be p and s, p and T or any convenient pair. It turns out in the present problem that pressure p and density ρ are the most convenient ones to designate as the independent properties. Then temperature T and entropy s become corresponding dependent properties which can be expressed in terms of p and ρ by means of the usual perfect gas relations. Of course, velocity components u and v must still be specified to complete the definition of the conditions at a given field point.

The above considerations indicate that the four variables u , v , p and ρ are both necessary and sufficient to fix conditions at any point in the field. Hence our mathematical model must provide the field equations and boundary conditions needed to fix the distributions of these four fundamental variables.

In this respect, what should be said about quantities like eddy viscosity ε , mean turbulent energy E , and possibly other variables that characterize the turbulence? For the purposes of the present analysis, these may all be regarded as secondary variables. They are fixed by the details of the postulated turbulence model whenever the distributions of the above four fundamental variables are specified. Or to put the matter more simply, it suffices in the present context to treat ε and E merely as known functions. This is permissible even if we choose not to specify at this point the precise nature of these functions.

In addition to the general approach outlined above, there are of course a wide variey of alternative analytical methods described in the current literature. For example, see references [32] through [45].

6. The Paradox of Five Equations in Four Unknowns

The preceding discussion has disclosed that, assuming ε and E to be known functions, our problem requires that we find the distribution over the flow field of the four fundamental variable u, v, p and ρ. Then all other secondary variables may be readily found from various auxiliary relations.

The fundamental physical laws that are at our disposal for the solution of this problem are summarized below. Each law provides a corresponding scalar equation.

1. Momentum equation, direction x.
2. Momentum equation, direction r.
3. First law of thermodynamics
4. Law of conservation of mass
5. Second law of thermodynamics

The details of these equations are given elsewhere and need not concern us here. It is pertinent to note, however, that the partial derivatives of highest order which occur in these equations are the derivatives of second order of all four independent variables. Using a subscript notation to indicate partial differentiation and referring to axisymmetric flow in cylindrical coordinates, we may write these derivatives as u_{xx}, u_{xr}, u_{rr}, v_{xx}, v_{xr}, v_{rr}, p_{xx}, p_{xr}, p_{rr}, ρ_{xx}, ρ_{xr}, ρ_{rr}. Note that there are twelve of these second order derivatives in all. Viewed in these terms, the problem is seen to be an extraordinarily complicated one.

The above discussion confronts us at once with a curious paradox. It would seem that we are required to solve five simultaneous second order partial differential equations in four unknowns! This certainly clashes with the ordinary concept that in order to ensure a determinate solution, the number of

equations and the number of unknowns should be equal. The above system of equations appears to be overdetermined and hence possibly inconsistent and insoluble.

It would seem that in order to define a determinate solution, it is necessary to drop one of the five governing equations. If so, which one should it be? There appears to be no satisfactory answer to this question because every one of the five makes a physical claim which cannot be denied.

Needless to say, this paradox created considerable consternation before it was finally resolved. But resolved it was, and in an entirely rational way as the reader will presently see. The nature of the resolution is in some respects surprising and amounts to a new insight concerning the basic mathematical character of compressible turbulent flow. At least the author knows of no publication other than the present report in which these novel features are disclosed and systematically explained.

In order to by-pass temporarily the paradox of five equations in four unknowns, it was decided to drop the second law of thermodynamics for the time being and to proceed with the solution of the remaining four equations in four unknowns. The hope was that at some point in the solution procedure an opportunity might still arise to reintroduce the temporarily neglected fifth equation and, in fact, this is how matters finally worked out.

The decision thus to by-pass initially the second law of thermodynamics rather than one of the other five basic equations was to a certain extent arbitrary. It was based mainly on convenience. However, since the temporarily discarded equation eventually reappears in the analysis, it becomes a matter of relatively minor importance which one of the five is treated in this way.

7. Nonturbulent Compressible Flows of Mixed Elliptical/Hyperbolic Type

In order to obtain a solution of our residual system of four basic equations in four unknowns, it is necessary to establish whether these equations are of hyperbolic, elliptic or mixed type. This is not a question of mere academic interest but an absolute necessity because the answer to this question fixes just how the boundary conditions must be posed in order to define a determinate solution.

To help put this question into its proper perspective, it is useful first to review the situation as it applies to the simpler problem of axisymmetric, nonturbulent compressible flow. It suffices here to consider irrotational motion for which a velocity potential ϕ exists. The governing equation may now be represented in the following somewhat generalized, quasi-linear form

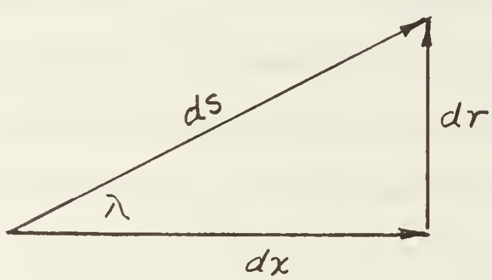
$$A\phi_{xx} + 2B\phi_{xr} + C\phi_{rr} = D \tag{7.1}$$

where coefficients A, B, C, D represent certain known functions of the velocities components ϕ_x and ϕ_r ; the exact nature of these functions will be specified a little later. They are also discussed by Shapiro [32].

Now consider the following auxiliary relations

$$\begin{aligned} d\phi_x &= \phi_{xx}\, dx + \phi_{xr}\, dr \\ d\phi_r &= \phi_{xr}\, dx + \phi_{rr}\, dr \end{aligned} \tag{7.2}$$

Let ds be a small displacement in the plane at angle λ as shown. Thus



$$\begin{aligned} dx &= ds \cos\lambda \\ dr &= ds \sin\lambda \end{aligned} \tag{7.3}$$

Now substituting Eqs. (7.3) into (7.2), then dividing through by ds and rearranging gives

$$\left(\frac{d\phi_x}{ds}\right) = \phi_{xs} = \phi_{xx}\cos\lambda + \phi_{xr}\sin\lambda$$
$$\left(\frac{d\phi_r}{ds}\right) = \phi_{xr} = \phi_{xr}\cos\lambda + \phi_{rr}\sin\lambda \tag{7.4}$$

Eqs. (7.1) and (7.4) may next be reassembled in matrix format as follows.

$$\begin{bmatrix} \cos\lambda & \sin\lambda & 0 \\ A & 2B & C \\ 0 & \cos\lambda & \sin\lambda \end{bmatrix} \begin{Bmatrix} \phi_{xx} \\ \phi_{xr} \\ \phi_{rr} \end{Bmatrix} = \begin{Bmatrix} \phi_{xs} \\ D \\ \phi_{xr} \end{Bmatrix} \tag{7.5}$$

In general the above equations can be inverted, that is, solved for ϕ_{xx}, ϕ_{xr} and ϕ_{rr}, provided only that the determinant of the array on the left is nonvanishing. Since some of the elements of the array are functions of λ, the determinant itself, let us denote it by symbol $D(\lambda)$, may in general also be a function of λ. The question therefore arises whether there exist any characteristic values of λ for which the determinant vanishes.

To see what this question implies, assume for a moment that there does exist a family of characteristic curves whose local direction at every point as defined by angle λ is such that $D(\lambda) = 0$ everywhere. It turns out that if we now try to determine the quantities ϕ_{xx}, ϕ_{xr}, ϕ_{rr} by attempting to invert Eqs. (7.5), and by evaluating the quantities ϕ_{xs} and ϕ_{rs} along such a characteristic line, the resulting "solutions" will simply assume the indeterminate forms $0/0$. What this means is that even though the first derivatives φ_x and φ_r are everywhere continuous, the second derivatives φ_{xx}, φ_{xr}, φ_{rr} are in general indeterminate and may therefore be discontinuous

along any such characteristic line. On the other hand if no characteristic directions or lines exist, the quantities φ_{xx} , φ_{xr} , φ_{rr} must remain determinate and continuous everywhere and the whole qualitative character of the solution is thereby radically changed.

Hence we ask whether there exist any real values of λ Such that

$$D(\lambda) = \begin{vmatrix} \cos\lambda & \sin\lambda & 0 \\ A & 2B & C \\ 0 & \cos\lambda & \sin\lambda \end{vmatrix} \tag{7.6}$$

$$= - A \sin^2\lambda + 2B \sin\lambda \cos\lambda - C \cos^2\lambda = 0$$

Dividing Eq. (7.6) through by $\cos^2\lambda$ and changing signs gives

$$A \tan^2\lambda - 2B \tan\lambda + C = 0 \tag{7.7}$$

Next solving Eq. (7.7) by the quadratic formula, we obtain the key result

$$\tan\lambda = \frac{B \pm \sqrt{B^2 - AC}}{A} \tag{7.8}$$

Eq. (7.8) reveals three possibilities:

1) At any point in the field for which $B^2 > AC$, there exist two real values of angle λ for which $D = 0$. The equation is said to be hyperbolic at such a point.

2) At any point in the field for which $B^2 \equiv AC$, there exists one real value of angle λ for which $D = 0$. The equation is said to be parabolic at such a point.

3) At any point for which $B^2 < AC$, there exist no real values of angle λ for which $D = 0$. The equation is said to be elliptical at such a point.

If the equation be hyperbolic over certain regions of the flow field and elliptical over other regions, it is said to be of mixed type. In such cases there will exist some boundary line which separates the hyperbolic and elliptical regions. For points which lie on this boundary, the equation will be parabolic.

One of the basic difficulties of solving equations of mixed type is that the boundary conditions of the problem must be specified differently around the elliptic and hyperbolic regions. Also, over the elliptic region, the basic equation may be formulated in terms of ordinary finite differences whereas over the hyperbolic region it must be formulated in terms of the method of characteristics. Worse still, the location of the boundary between the two regions is itself initially unknown. For these reasons problems of mixed type often prove to be hopelessly intractable. Nevertheless some problems of mixed type can still be solved. For example, see Jameson [48].

Further physical insight can be gained by studying the specific nature of the coefficients A , B , C , D for the present problem. Let symbol c denote the local velocity of sound at an arbitrary point and let symbol M denote the corresponding local Mach number. Also let

$$
\begin{aligned}
M_x &= \frac{\varphi_x}{c} \\
M_r &= \frac{\varphi_r}{c} \qquad (7.9) \\
M^2 &= M_x^{\,2} + M_r^{\,2}
\end{aligned}
$$

By a minor modification of the analysis of Shapiro [32], it can be shown that for our present problem

$$\begin{aligned} A &= 1 - M_x^2 \\ B &= - M_x M_r \\ C &= 1 - M_r^2 \\ D &= - \varphi_r / r \end{aligned} \tag{7.10}$$

Upon substituting Eqs. (7.10) into (7.8) and simplifying, we obtain

$$\tan\lambda = \frac{- M_x M_r \pm \sqrt{M^2 - 1}}{(1-M_x^2)} \tag{7.11}$$

This now reveals clearly that the basic equation is hyperbolic over regions where the flow is locally supersonic, and elliptical over regions where the flow is locally subsonic. This is a well known result.

It is also well known that if the basic equation be of second order and everywhere elliptic, that is, if the flow field be everywhere subsonic, the boundary conditions which are both necessary and sufficient to define ϕ uniquely over a given region may be summarized as follows. Let C denote an arbitrary closed contour which encloses the region of interest. Let $\vec{n}$ be an outward unit vector normal to contour C at a general point. Then to fix the distribution of ϕ uniquely over the region of interest, it is necessary to specify at every point of contour C the value of either one of two quantities. The first is simply the value of ϕ itself. The second is the normal derivative of ϕ, that is, the quantity

$$\vec{n} \cdot \nabla\phi = \left(\frac{\partial\phi}{\partial n}\right) = \phi_n \tag{7.12}$$

Under certain special conditions the function ϕ can be multi-valued in the region of interest. Under these circumstances it becomes necessary to specify also the value of the circulation integral around contour C .

If the function ϕ is single valued, however, the circulation integral equals zero. Fortunately, the multi-valued solution is of little interest in the present discussion so that it is not necessary to elaborate on the details of this case.

We shall not attempt to summarize the corresponding rules for the boundary conditions if the basic equation be hyperbolic, that is, if the flow field be everywhere supersonic, because it is not possible to do so concisely and because these details are not really needed here. It should be pointed out, however, that the boundary conditions in this case cannot in general be arbitrarily specified completely around a closed contour. Conditions over certain portions of the contour must now be left free to be established by the details of the solution itself.

These various mathematical conclusions all have a straightforward physical interpretation which stems from the simple fact that any small disturbance in the flow field is propagated in all directions at sonic velocity with respect to the fluid itself. In subsonic flow such signals therefore propagate throughout the entire flow field so that conditions at every point are influenced by conditions at every other point. In supersonic flow, on the other hand, signals cannot propagate upstream and certain regions of the flow field are totally unaffected by what happens over certain other regions.

The question that now arises is this: To what extent, if any, do the foregoing conditions, which are know to apply when the flow is treated as compressible but nonturbulent, continue to apply when the flow is treated as both compressible <u>and</u> turbulent?

Presumably the introduction of turbulence effects should not greatly influence the rate of propagation of physical signals as outlined above. From this consideration one might suppose - at least this author did - that the

above picture of the mixed elliptical/hyperbolic character of the solution would for the most part continue to prevail also for the turbulent case. In fact this opinion was in itself a rather discouraging element in the situation as it tended to dampen hopes for developing a tractable solution.

On the other hand a mere opinion of this kind does not in itself provide an adequate basis for actually specifying the necessary and sufficient boundary conditions which are required for our more generalized equations. Note that the foregoing picture is drawn from the classical analysis of a situation which is governed by a single second order equation in the single unknown ϕ . Recall, however, that our more generalized problem as developed to this point in the argument, involves the solution of four simultaneous second order equations in the four unknowns u , v , p and ρ . Also recall that we have an as yet unused fifth equation still lurking in the background. The elliptic/parabolic/hyperbolic characteristics of our solution must now be deduced rigorously from the four equations actually employed; these features cannot be adequately inferred by mere analogy with the simpler classical case discussed at length in this section.

Such a rigorous and independent analysis was indeed carried out. It proved to be a lengthy and arduous task. The high points of this analysis are summarized in the next section. Let it suffice here to say that the results proved to be a stunning surprise! The classical picture of a problem of mixed elliptic/hyperbolic type as outlined in this section was found to undergo a radical transformation. At first this result was very puzzling. In due course, however, it was found to provide just the conditions needed to make our hitherto unutilized fifth equation again relevant and necessary to the solution. Thus the story has a happy ending.

Further discussion of equations of elliptical, parabolic, hyperbolic and mixed type may be found in references [46] through [55].

8. Classification of Basic Equations for Turbulent Compressible Flow

In the previous section we considered the non-turbulent case in which the flow is governed by a single equation of second order in the three second order derivatives ϕ_{xx} , ϕ_{xr} and ϕ_{rr} . The classification of this equation as being of elliptic, parabolic or hyperbolic type at an arbitrary point was seen to hinge on the possible vanishing of a certain three by three characteristic determinant. This determinant arises from the fact that the governing equation is augmented by two auxiliary relations.

In this section we wish to consider broadly the generalization of the above analysis to the turbulent case in which the flow is governed by four simultaneous second order equations in the twelve second order derivatives u_{xx} , u_{xr} , u_{rr} , v_{xx} , v_{xr} , v_{rr} , p_{xx} , p_{xr} , p_{rr} , ρ_{xx} , ρ_{xr} , ρ_{rr} . Once again, each of the four basic equations is augmented by two auxiliary relations. Hence we must now deal with twelve simultaneous equations in twelve unknowns. Classification of the basic equations as being of elliptic, parabolic or hyperbolic type at an arbitrary point again hinges on the possible vanishing of a certain characteristic determinant, this time of dimension twelve by twelve.

The details of this determinant are fully explained in a later section of this report and need not concern us here. However, the implications of dealing with a determinant of this size are startling and amusing and worth pointing out for these reasons. The general result obtained, and the conclusions that follow from this result are also quite radical and require appropriate interpretation. These are the general aspects which are discussed in this section.

It can be shown that the full expansion of an n by n determinant, none of whose individual elements is zero, amounts to $n!$ terms in all, each term of which is the product of n factors. The extraordinary implications of this rule are seldom appreciated. They can best be illustrated by the following two examples. The full expansion of a 3 by 3 determinant amounts to $3! = 6$ terms in all, each of which is the product of 3 factors. On the other hand the full expansion of a 12 by 12 determinant amounts to the staggering total of $12! = 479{,}001{,}600$ terms in all, each of which is the product of 12 factors!

Fortunately, the actual 12 by 12 determinant of interest in the present case is rather sparse. Only 36 of its 144 elements are non-zero. Consequently, its evaluation is not nearly as hopeless a task as the above figures might suggest. Nevertheless, it is still a formidable undertaking which runs to about two dozen pages of algebra. Note too that the expansion required is algebraic, not merely numerical. This greatly increases the labor involved.

The final result obtained from all of this strenuous effort turns out to amazingly simple. The characteristic determinant is found to be

$$D = \frac{4}{3} \frac{1}{(\gamma-1)} \tag{8.1}$$

where γ represents the ratio of specific heats.

A very surprising feature of the above determinant is that it turns out to be independent of angle λ . This is unexpected because many of the individual elements of the original 12 by 12 array are themselves functions of λ . In the process of expanding the determinant, various complicated intermediate functions of λ are formed. In the final expansion, however, all of these complex intermediate functions cancel out of the result. It is quite remarkable.

Another surprising feature of the characteristic determinant is that it is entirely independent of the local Mach number. In this respect the present result for turbulent compressible flow is wholly unlike the classical result for nonturbulent compressible flow as analyzed in the previous section.

The most important conclusion that follows from Eq. (8.1) is that the four governing equations for this case are always of elliptical type! Notice that this conclusion holds regardless whether the flow is locally subsonic or supersonic.

In a way this result which was initially such a surprise also turned out to be a vast relief. The reason is that it was the first solid indication that the present problem might be solved in a manner that avoids the extreme difficulties that plague problems of the mixed elliptic/hyperbolic type.

It is now an apparently simple matter to formulate the nature of the boundary conditions which must be specified and which are both necessary and sufficient to fix the detailed distributions of the four variables u, v, p, ρ over some arbitrary region of interest. Let C denote some arbitrary closed contour which enclosed the region. It is now necessary to specify at every point of contour C the values of four quantities. These are the quantities u or $(\frac{\partial u}{\partial n})$, v or $(\frac{\partial v}{\partial n})$, p or $(\frac{\partial p}{\partial n})$ and ρ or $(\frac{\partial \rho}{\partial n})$.

At first glance this seems simple enough. Unfortunately, it is all too simple for it turns out on more detailed examination that we do not seem to have sufficient information to fix the above four boundary conditions all the way around contour C! The nature of these further difficulties, and the method that was finally developed for overcoming them, is the subject of the next section.

9. Boundary Conditions for the Elliptic and for the Piecewise Parabolic Cases

In this section we analyze the problem of defining adequate boundary conditions for our problem of a ducted jet. For definiteness, we consider the slightly simplified configuration indicated schematically in Fig. 9.1. Once the basic principles are clearly established, however, the extension to more complex geometries should offer no fundamental difficulties.

We wish to calculate flow conditions over a region such as that bounded by the closed contour AA'D'DA . Because of symmetry there is no need to consider the region below the x axis in the figure. The diagram depicts a supersonic jet entering across portion AA" of the boundary. By a minor change we could make the jet at AA" sonic if desired. After expanding, the jet discharges across exit station DD' .

Since the final calculations are done by finite differences, it is expedient to define the boundary conditions in corresponding terms. A square computational grid of mesh size a is used. The position of a particular point in this grid is indicated by the indices m and k as shown.

For reasons which will soon appear, it is advantageous to divide the domain of analysis into sub-regions or cells of axial length $5a$ as shown. The first such cell is bounded by contour AA'B'BA , the next by contour BB'C'CB , and so on.

Consider the first cell AA'B'BA . According to the analysis of the previous section, the four basic equations permit us to solve for the values of the four variables u , v , p and ρ at all interior points of this cell once conditions are properly specified for the points which lie along the boundary. Four boundary conditions must be specified for each boundary point. The simplest situation is one in which the actual numerical values of u , v , p and ρ

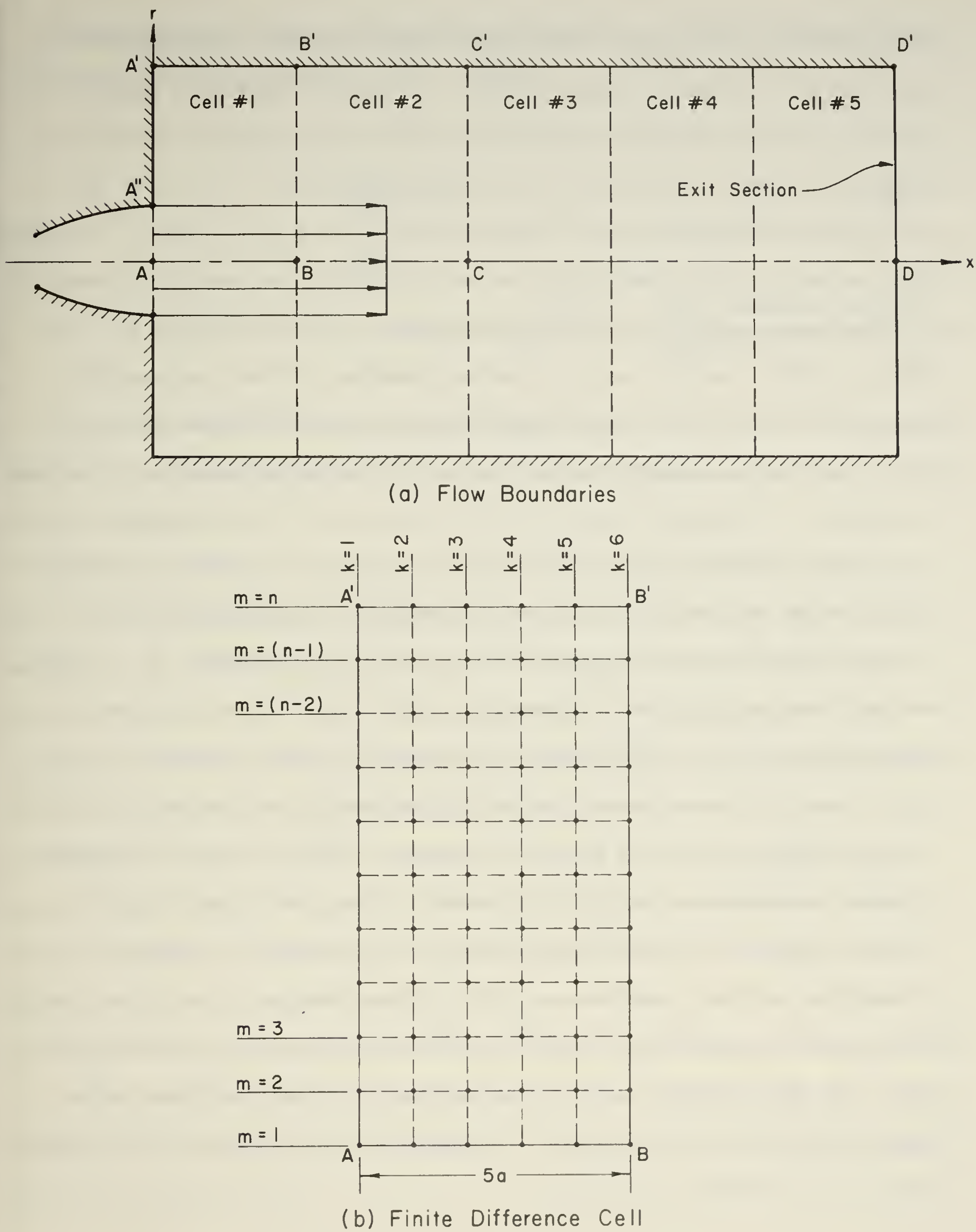


FIG. 9.1 FINITE DIFFERENCE MESH FOR DUCTED JET PROBLEM

are stipulated at the point. The boundary conditions need not necessarily be restricted in so severe a manner, however. It suffices merely to have four applicable finite difference equations for each boundary point. These may be expressed in terms of the four initially unknown values of u , v , p and ρ at that point and usually of the unknown values of u , v , p and ρ at nearby points as well. Under these circumstances it might not be possible to establish all unknowns along the boundary before proceeding with the solution for the interior points. In that case the boundary unknowns and the corresponding boundary equations must be incorporated into the overall solution matrix. As long as this overal problem incorporates as many independent equations as there are unknowns, a complete solution fom all unknowns can still be found.

Let us now examine the various portions of the cell boundary to ascertain in more detail the nature of the boundary conditions that apply to each.

The simplest situation occurs at points that lie on segment AA'' of the boundary, that is, along the cross-section of the entering jet. We are at liberty to specify the actual numerical values of all four variables u , v , p , ρ at each of these points. If the jet is parallel we must set $v = 0$.

An anomaly may arise at the edge of the jet, point A'' , if the velocity is treated as discontinuous at this location. Thus, just above point A'' u is zero, while just below it equals the full jet velocity. No finite difference scheme can accommodate a double valued function of this kind. Perhaps the simplest remedy is to take the effective value of u at this point as equal to the arithmetic mean of the two limiting values mentioned above. Another possibility is to position the edge of the jet so that it falls midway between two successive grid points.

It can be shown that the following four boundary conditions apply at each point along the solid boundary A'A' , namely,

$$
\begin{aligned}
u &= 0 \\
\left(\frac{\partial u}{\partial t}\right) &= 0 \\
\left(\frac{\partial T}{\partial x}\right) &= 0 \\
v &\doteq \pm\, 11.0 \sqrt{\frac{|\tau_{xr}|}{\rho}} \qquad \left\{\begin{array}{l}\text{The algebraic sign of} \\ v \text{ must agree with} \\ \text{that of } \tau_{xr} \text{ .}\end{array}\right\}
\end{aligned}
\tag{9.1}
$$

The first two of Eqs. (9.1) clearly express the condition of no flow through the wall. The third expression is the consequence of imposing zero heat transfer normal to the wall.

The fourth of Eqs. (.1) requires more extended comment. It is well known that at any fixed boundary in turbulent flow there exists a very thin viscous sublayer. The tangential velocity component changes very rapidly from zero at the wall itself to some finite value at the edge of the sublayer where the latter meets the main turbulent flow. The tangential velocity at this location is only moderately smaller than that in the main turbulent region itself. It is possible to approximate the tangential velocity at the edge of the viscous sublayer analytically by means of a so-called wall function and that is what we have done above. This procedure avoids the difficulty of trying to resolve the extremely rapid changes through the viscous sublayer by finite differences; the regular computational grid is far too coarse for this purpose. The quantity τ_{xr} in the fourth of Eqs. (9.1) represents the shear stress at the wall. The derivation of this approximate expression is given in section 17 of this report and is not repeated here.

Eqs. (9.1) can be readily rewritten in terms of the primary variables u, v, p, ρ and then further simplified on the basis of plausible physical arguments. This too is explained in section 17 and is not repeated here. The significant point to be made here is simply that we have identified the four basic physical constraints that fix the four boundary equations which apply at each point along a stationary wall.

Of course analogous conditions apply along the outer wall segment A'B'. Thus

$$\begin{aligned} &v = 0 \\ &\left(\frac{\partial v}{\partial t}\right) = 0 \\ &\left(\frac{\partial T}{\partial r}\right) = 0 \\ &u \doteq \pm\, 11.0 \sqrt{\frac{|\tau_{xr}|}{\rho}} \qquad \left\{\begin{array}{l}\text{The algebraic sign of} \\ u \text{ must be opposite to} \\ \text{that of } \tau_{xr}\text{ .}\end{array}\right\} \end{aligned} \tag{9.2}$$

Next consider conditions along segment AB of the x axis. Actually the points that lie along this line may be treated much like those in the interior region. That is to say the four basic equations may be applied at these points. The only difference is that these equations will reduce somewhat because of the symmetry conditions that prevail along the axis. The conditions are simply that

$$v = 0$$

and

$$\frac{\partial}{\partial r} = 0 \tag{9.3}$$

Examination of the four basic equations for this case reveals that the momentum equation in direction r is now satisfied identically. The other three basic equations then provide the conditions necessary to fix the values of u, p and ρ for points on the axis.

Finally we consider the exit cross-section of the cell, boundary segment B'B . Boundary conditions for the portion BAA'B' of the enclosing contour have been established in the foregoing discussion. However, the solution for the values of u , v , p , ρ at the interior points cannot be completed until the values of u , v , p , ρ are specified also for the points along the final segment B'B of the cell contour. How shall these initially unknown parameters be prescribed?

Notice that since the four basic equations comprise an elliptical system, a solution which rigorously satisfies all four of these equations at all interior points can always be found irrespective of how the above unknown parameters along the exit station B'B happen to be assigned. Thus our analysis leaves us at this point not with a unique solution but rather with a large family of theoretically possible solutions. The essential differences between individual solutions of this family lie in how the boundary conditions happen to be assigned across station B'B for each case. However, if the final solution of the problem is unique, there must exist some rational basis for prescribing these as yet undetermined boundary conditions across B'B in a corresponding unique fashion.

In dealing with segments BA , AA" , and A'B' of the boundary, it was possible in every case to base the respective boundary conditions on physical considerations of a very obvious and straightforward kind. No corresponding elementary considerations seem to apply to portion B'B of the enclosing boundary.

The question arises whether the above apparent indeterminacy in the boundary conditions might be resolved by enlarging the domain of analysis to include the entire region inside contour AA'D'DA . The idea here is that conditions far downstream across section D'D should be far simpler to prescribe correctly than those across any interior section such as B'B . Proceeding on this basis

we may postulate, for example, that $v = 0$ along $D'D$, at least to a close approximation. This seems quite plausible. On the other hand all efforts to find equally plausible and satisfactory assumptions for the other three conditions along $D'D$, as required to define a unique solution, have proved unsuccessful.

Unfortunately the flow field far downstream, instead of converging asymptotically to some limit characterized by negligible rates of change of key parameters, eventually approaches a choking condition characterized by extremely large rates of change. The melancholy conclusion must be drawn that it is not possible to stipulate conditions far downstream in advance. Presumably these conditions, instead of fixing the solution, are themselves somehow fixed by the solution.

It was eventually realized that the resolution of the above seeming impasse can only be accomplished by utilizing the mathematical resource which we were previously compelled to lay aside temporarily. This is the fifth of the five basic equations, namely, the second law of thermodynamics. Of the great multiplicity of theoretical solutions, all of which satisfy the first four basic equations equally well, not all are of equal merit with respect to the second law. In fact we may use the second law as a kind of mathematical filter to cull out all but one of these many solutions, retaining only that one solution which satisfies not only the four basic equations but also the second law. In that way we finally obtain a unique solution which does in fact satisfy five basic equations in four unknowns!

To see more clearly how this can be done, consider again the cell domain enclosed within contour $BAA'B'B$. In particular, consider the row of points at the fixed radial station denoted by index m. The last point in this row

is at axial station $k = 6$. There are four initially unknown boundary conditions that must be stipulated at this point, namely, u_{m6} , v_{m6} , p_{m6} , ρ_{m6} .

The second law requires that a certain set of terms must sum to zero at each interior point in the region. If the second law is not properly satisfied at a particular point (m, k) , these terms will sum not to zero but to some finite magnitude, call it z_{mk} . Thus we now require that parameters $u_{m,6}$, v_{m6} , p_{m6} , ρ_{m6} be so chosen as to give

$$\begin{aligned} &z_{mk} = 0 \\ &k = 2,3,4,5 \\ &m = 1,2,3, \ldots. (n-1) \end{aligned} \tag{9.4}$$

Notice that Eqs. (9.4) provide four equations in four unknowns for each station m . Of course these equations must be satisfied for all $(n-1)$ interior stations m as indicated. Thus Eqs. (9.4) compress $4(n-1)$ equations in $4(n-1)$ unknowns.

In principle Eqs. (9.4), when combined with the four original basic equations at each interior point and with the four boundary conditions at each point along the open contour BAA'B' , just suffice to fix the complete solution on and within the closed contour BAA'B'B , Moreover, this solution is now unique; there are no further undetermined parameters.

It should now be clear why each cell was specified to be of axial length 5a. If it were any longer, there would not be a sufficient number of free parameters available at the exit station to satisfy the second law at all interior points. If it were any shorter, there would not be a sufficient number of interior points to utilize all of the degrees of freedom available across the exit station.

It is evident that essentially the same solution may then be applied to the next cell, that is, to the region enclosed within contour CBB'C'C . If anything, this second cell should be slightly simpler to calculate than the first because the conditions across section B'B are known in a simpler form than those which prevail across section A'A . All succeeding cells will also be simpler in the same sense.

We see that the solution can be marched downstream, cell by cell, as far as desired. Such a unidirectional progression is characteristic of equations which are everywhere parabolic. The present solution proceeds downstream in finite increments of length 5a. Hence it may be characterized as being "piecewise parabolic". The successive cross-sections A'A , B'B , C'C , and so on play the role of characteristics.

It is a quite extraordinary fact that we have now demonstrated, namely, that the basic equations of turbulent compressible flow are in the final analysis neither elliptical, hyperbolic, nor mixed, but are rather piecewise parabolic. It is remarkable too that this conclusion applies whether the flow happens to be subsonic, supersonic or both. This implies also that a method of solution exists which, while quite complex, is nevertheless substantially simpler than that which characterizes equations of mixed elliptic/hyperbolic type.

In this one respect the introduction of turbulence effects into the mathematical model actually simplifies the analysis.

10. Solution for Cell Exit Conditions by Relaxation

In later sections of this report, a set of four basic equations is derived which suffice to fix the four independent variables u , v , p , ρ at each of the interior points of a cell. Let the number of radial stations starting at

the axis, $r = 0$, and ending at the outer wall, $r = R$, be denoted by index $m = 1,2,3, \ldots n$. For the first cell of axial length 5a, let the axial stations be denoted by index $k = 1,2,3,4,5,6$. If we include the points along the axis among the interior points, the total number of interior points is

$$N = 4\ (n - 1) \tag{10.1}$$

It is useful to denote these interior points by index i , so that

$$i = 1,2,3, \ldots N \tag{10.2}$$

The theory also develops a fifth equation which expresses the amount z_i by which the second law is not satisfied at the ith interior point.

We seek a solution such that

$$z_i = 0$$

$$i = 1,2,3 \ldots N \tag{10.3}$$

It proves advantageous, however, to satisfy Eqs. (10.3) indirectly by imposing the alternative but equivalent restriction that

$$I = \sum_{i=1}^{N} r_i\, z_i^{\,2} = \text{A Minimum} \tag{10.4}$$

The following notation will also prove convenient at this stage. Let the various variables u , v , p , ρ at points along the exit station $k = 6$ be denoted by the generalized symbol α_j , where $j = 1,2,3, \ldots N$, according to the following scheme.

$$\begin{array}{llll} u_{1,6} = \alpha_1 & u_{2,6} = \alpha_5 & --- & u_{(n-1),6} = \alpha_{(N-3)} \\ v_{1,6} = \alpha_2 & v_{2,6} = \alpha_6 & --- & v_{(n-1),6} = \alpha_{(N-2)} \\ p_{1,6} = \alpha_3 & p_{2,6} = \alpha_7 & --- & p_{(n-1),6} = \alpha_{(N-1)} \\ \rho_{1,6} = \alpha_4 & \rho_{2,6} = \alpha_8 & --- & \rho_{(n-1),6} = \alpha_N \end{array} \tag{10.5}$$

Now consider the small change

$$\delta z_i \doteq \left(\frac{\partial z_i}{\partial \alpha_j}\right) \delta \alpha_j \tag{10.6}$$

that occurs in the solution for z at the ith interior point as a result of an arbitrary small change $\delta\alpha_j$ in the single arbitrary boundary parameter α_j. The error term I is therefore changed accordingly. The resulting new value, call it I', is now given by the approximate expression

$$I' \doteq \sum_{i=1}^{N} r_i \left[z_i + \left(\frac{\partial z_i}{\partial \alpha_j}\right) \delta \alpha_j\right]^2$$

$$\doteq \sum_{i=1}^{N} r_i \left[z_i^2 + 2z_i \left(\frac{\partial z_i}{\partial \alpha_j}\right) \delta \alpha_j + \left(\frac{\partial z_i}{\partial \alpha_j}\right)^2 (\delta \alpha_j)^2\right]$$

$$\doteq \sum_{i=1}^{N} r_i \, z_i^2 + \delta \alpha_j \, 2 \sum_{i=1}^{N} r_i z_i \left(\frac{\partial z_i}{\partial \alpha_j}\right) + (\delta \alpha_j)^2 \sum_{i=1}^{N} r_i \left(\frac{\partial z_i}{\partial \alpha_j}\right)^2$$

$$\doteq \text{A Minimum} \tag{10.7}$$

Let us now choose the as yet undetermined parameter $\delta\alpha_j$ in such a way as to minimize the above function. The minimum occurs when the corresponding derivative vanishes, that is, when

$$\frac{\partial I'}{\partial(\delta\alpha_j)} = 0 + 2 \sum_{i=1}^{N} r_i z_i \left(\frac{\partial z_i}{\partial \alpha_j}\right) + 2(\delta\alpha_j) \sum_{i=1}^{N} r_i \left(\frac{\partial z_i}{\partial \alpha_j}\right)^2 = 0 \tag{10.8}$$

The solution of this equation now gives the required change in the form

$$(\delta\alpha_j) = - \frac{\sum_{i=1}^{N} r_i z_i \left(\frac{\partial z_i}{\partial \alpha_j}\right)}{\sum_{i=1}^{N} r_i \left(\frac{\partial z_i}{\partial \alpha_j}\right)^2} \tag{10.9}$$

Successive applications of Eqs. (10.9) for $j = 1,2,3, \ldots N$, and repetition of this cycle of calculations as many times as necessary, will ultimately reduce the error term I below any preassigned bound and thus provide a solution of any desired accuracy.

While the above solution necessarily converges, the calculations are inherently very lengthy. This follows from the fact that for each small change $\delta\alpha_j$ in an arbitrary one of the N parameters, a complete solution for the resultant changes $(\frac{\partial z_i}{\partial \alpha_j})$ must be worked out over the entire field, $i = 1,2,3, \ldots N$. It seems possible that such changes might prove to be local in character, so that effects some distance away from the boundary point in question can be neglected. If so, this fact should permit the calculations to be very substantially shortened. The details cannot be settled here but must be worked out in connection with the development of the actual computer code.

11. Basis for the Further Analysis

In the present context, cylindrical coordinates x, r, Θ represent the natural choice. All subsequent relations are expressed either in this specific coordinate system or else in generalized vector notation which applies in any coordinate system.

All mean flow quantities are taken as axi-symmetric and the peripheral component of the mean velocity is taken as zero everywhere.

While the ultimate solution that is sought is for the case of steady mean flow, the basic equations are initially developed for the more general case of unsteady mean flow. By holding the pertinent boundary conditions constant, we can follow the unsteady motion through time numerically until the required steady state finally develops. This technique is in some respects simpler than attempting to find the final steady state directly from the steady state equations themselves.

The fluid is treated as a perfect gas with constant specific heats. Adiabatic conditions are assumed at all solid boundaries.

The net turbulent transport of momentum is assumed to be expressible in terms of an appropriate eddy viscosity ε, and the net turbulent transport of mass, energy or entropy is assumed to be expressible in terms of a related transport coefficient $\kappa\varepsilon$ as explained earlier.

The three velocity components and the various scalar properties of the fluid such as its density, pressure, temperature and so on are expressed according to the following format.

$$
\begin{aligned}
u''(x,r,\theta,t) &= u(x,r,t) + u'(x,r,\theta,t) \\
v''(x,r,\theta,t) &= v(x,r,t) + v'(x,r,\theta,t) \\
w''(x,r,\theta,t) &= 0 + w'(x,r,\theta,t) \\
\rho''(x,r,\theta,t) &= \rho(x,r,t) + \rho'(x,r,\theta,t) \\
p''(x,r,\theta,t) &= p(x,r,t) + p'(x,r,\theta,t) \\
T''(x,r,\theta,t) &= T(x,r,t) + T'(x,r,\theta,t)
\end{aligned}
\qquad (11.1)
$$

etc.

In these expressions the unprimed quantities represent the mean flow effects whose distributions it is required to calculate. The single primed quantities represent the corresponding turbulent fluctuations. The double

primed quantities represent the corresponding instantaneous resultant variables.

It is necessary to simplify the various basic equations by a suitable process of averaging. In the present case all mean flow quantities are unsteady and the appropriate average is the ensemble average. We denote the ensemble average of any fluctuating quantity in the usual way by means of an overbar.

Upon averaging Eqs. (11.1) in this way we obtain the following relations.

$$\begin{aligned}
\overline{u''} &= u\ (x,r,t) & \overline{u'} &= 0 \\
\overline{v''} &= v\ (x,r,t) & \overline{v'} &= 0 \\
\overline{w''} &= 0 & \overline{w'} &= 0 \\
\overline{\rho''} &= \rho\ (x,r,t) & \overline{\rho'} &= 0 \\
\overline{p''} &= p\ (x,r,t) & \overline{p'} &= 0 \\
\overline{T''} &= T\ (x,r,t) & \overline{T'} &= 0 \\
& & &\text{etc.}
\end{aligned} \tag{11.2}$$

Eqs. (11.2) show that the ensemble average of any single primed quantity is zero by definition. On the other hand the average value of the product of two or more singly primed quantities is not in general zero. Thus quantities like $\overline{u'v'}$, $\overline{\rho'u'}$, $\overline{\rho'u'v'}$ etc. do not vanish from the basic equations and must therefore be adequately accounted for in the final mathematical model.

A number of other mean and fluctuating quantities relating to the turbulent energy and to the turbulent transport of mass, energy and entropy are also involved in the development. These quantities are introduced and defined as the need for them arises in the subsequent analysis.

When dealing with the equations on a vectorial basis, one quantity that frequently arises is the instantaneous velocity vector $\vec{V}''$, namely,

$$\vec{V}'' = \vec{V} + \vec{V}' = \vec{e}_x u'' + \vec{e}_r v'' + \vec{e}_\theta w'' = \vec{e}_x\ (u+u') + \vec{e}_r\ (v+v') + \vec{e}_\theta\ (0+w') \tag{11.3}$$

12. Continuity Equation

The law of the conservation of matter, upon being ensemble averaged, requires that at every point in the flow field

$$\left(\frac{\partial \overline{\rho''}}{\partial t}\right) = \left(\frac{\partial \rho}{\partial t}\right) = - \nabla \cdot \ \overline{(\rho''\vec{V}'')} \tag{12.1}$$

Consider the quantity

$$\overline{\rho''\vec{V}''} = \overline{(\rho + \rho')(\vec{V} + \vec{V}')} = \rho\vec{V} + \overline{\rho'\vec{V}'} \tag{12.2}$$

The last term in Eq. (12.2) may be expressed in terms of a transport coefficient as follows

$$\begin{aligned} \overline{\rho'\vec{V}'} &= \vec{e}_x \ \overline{\rho'u'} + \vec{e}_r \ \overline{\rho'v'} + \vec{e}_\theta \ \overline{\rho'w'} \\ &= - \kappa\varepsilon \left[\vec{e}_x \left(\frac{\partial \rho}{\partial x}\right) + \vec{e}_r \left(\frac{\partial \rho}{\partial r}\right) + 0\right] \qquad (12.3) \\ &= - \kappa\varepsilon\nabla\rho \end{aligned}$$

Combining the foregoing relations and simplifying gives finally

$$\left(\frac{\partial \rho}{\partial t}\right) = - \nabla \cdot \ [\rho\vec{V} - \kappa\varepsilon\nabla\rho] \tag{12.4}$$

When expressed in cylindrical coordinates this becomes

$$\left(\frac{\partial \rho}{\partial t}\right) = - \frac{\partial}{\partial x} \left[\rho u - \kappa\varepsilon \left(\frac{\partial \rho}{\partial x}\right)\right] - \frac{1}{r}\frac{\partial}{\partial r} \left[r \left(\rho v - \kappa\varepsilon \frac{\partial \rho}{\partial r}\right)\right] \tag{12.5}$$

Eq. (12.5) represents one of the fundamental equations of the final mathematical model.

13. Momentum Equations

In vector notation the equation of motion, including effects of turbulent fluctuations, may be written

$$\left(\frac{\partial \vec{V}''}{\partial t}\right) + \vec{V}''\cdot\nabla V'' = -\frac{\nabla p''}{\rho''} \qquad (13.1)$$

This relation ignores gravity forces which are not significant in the present context, and viscous forces which are negligible everywhere except in the viscous sub-layer along a fixed boundary. Such wall regions will be considered separately later.

The left side of Eq. (13.1) represents the acceleration of a fluid particle. The right side represents the net force per unit mass acting on the particle.

The flow field also satisfies the continuity equation, namely,

$$\left(\frac{\partial \rho''}{\partial t}\right) + \nabla\cdot(\rho''\vec{V}'') = 0 \qquad (13.2)$$

Multiplying Eq. (13.1) by ρ'' , Eq. (13.2) by $\vec{V}''$, then adding and regrouping terms gives

$$\left[\rho''\left(\frac{\partial \vec{V}''}{\partial t}\right) + \left(\frac{\partial \rho''}{\partial t}\right)\ \vec{V}''\right] + \left[\rho\vec{V}''\cdot\nabla\vec{V}'' + \nabla\cdot\ (\rho''\vec{V}'')\ \vec{V}''\right] = -\nabla p'' \qquad (13.3)$$

This may be condensed to the form

$$\frac{\partial}{\partial t}\ (\rho''\vec{V}'') + \nabla\cdot\ (\rho\vec{V}''\ \vec{V}'') = -\nabla p'' \qquad (13.4)$$

Upon ensemble averaging this equation we obtain the important result

$$\frac{\partial}{\partial t}\ \overline{(\rho''\vec{V}'')} + \nabla\cdot\ \overline{(\rho''\vec{V}''\ \vec{V}'')} = -\nabla p \qquad (13.5)$$

By recalling Eq. (12.2) we note that the first term in Eq. (13.5) can be reduced to the form

$$\frac{\partial}{\partial t}\,\overline{(\rho''\ \vec{V}'')} = \frac{\partial}{\partial t}\,[\rho\vec{V} + \overline{\rho'\vec{V}'}] \tag{13.6}$$

The components of this expression in cylindrical coordinates are

$$\begin{aligned} \frac{\partial}{\partial t}\,(\overline{\rho''u''}) &= \frac{\partial}{\partial t}\,[\rho u + \overline{\rho'u'}] \\ \frac{\partial}{\partial t}\,(\overline{\rho''v''}) &= \frac{\partial}{\partial t}\,[\rho v + \overline{\rho'v'}] \end{aligned} \tag{13.7}$$

The next step is to expand the second term of Eq. (13.5) in cylindrical coordinates. Thus

$$\begin{aligned} &\nabla\cdot\ \overline{(\rho''\vec{V}''\vec{V}'')} \\ &= \left\{\frac{\partial}{\partial x}\ \overline{[\rho''u''\ (\vec{e}_x\ u'' + \vec{e}_r\ v'' + \vec{e}_\theta\ w'')]}\right. \\ &\quad + \frac{1}{r}\frac{\partial}{\partial r}\ \overline{[r\ \rho''v''\ (\vec{e}_x\ u'' + \vec{e}_r\ v'' + \vec{e}_\theta\ w'')]} \\ &\quad \left. + \frac{1}{r}\frac{\partial}{\partial \theta}\ \overline{[\rho''w''\ (\vec{e}_x\ u'' + \vec{e}_r\ v'' + \vec{e}_\theta\ w'')]}\right\} \end{aligned} \tag{13.8}$$

Notice that although $\vec{e}_x$ is constant, unit vectors $\vec{e}_r$ and $\vec{e}_\theta$ are functions of θ such that

$$\left(\frac{\partial\vec{e}_r}{\partial\theta}\right) = +\ \vec{e}_\theta \qquad \left(\frac{\partial\vec{e}_\theta}{\partial\theta}\right) = -\ \vec{e}_r \tag{13.9}$$

Upon expanding Eq. (13.8), making use of Eq. (13.9), and noting that of the eleven resulting terms, six vanish by reason of symmetry, we obtain

$$\begin{aligned} &\nabla\cdot\ \overline{(\rho''\vec{V}''\vec{V}'')} \\ &= \vec{e}_x \left\{\frac{\partial}{\partial x}\ [\overline{\rho''u''^2}] + \frac{1}{r}\frac{\partial}{\partial r}\ [r\ \overline{\rho''v''u''}]\right\} \\ &+ \vec{e}_r \left\{\frac{\partial}{\partial x}\ [\overline{\rho''u''v''}] + \frac{1}{r}\frac{\partial}{\partial r}\ [r\ \overline{\rho''v''^2}] - \left[\frac{\overline{\rho''w''^2}}{r}\right]\right\} \end{aligned} \tag{13.10}$$

By making use of Eqs. (13.7) and (13.10), we can now reduce Eq. (13.5) to the following pair of momentum transport equations in cylindrical coordinates, namely,

$$\frac{\partial}{\partial t}\left[\rho u + \overline{\rho' u'}\right] = -\left(\frac{\partial p}{\partial x}\right) - \frac{\partial}{\partial x}\left[\overline{\rho'' u''^2}\right] - \frac{1}{r}\frac{\partial}{\partial r}\left[r\,\overline{\rho'' v'' u''}\right]$$

$$\frac{\partial}{\partial t}\left[\rho v + \overline{\rho' v'}\right] = -\left(\frac{\partial p}{\partial r}\right) - \frac{\partial}{\partial x}\left[\overline{\rho'' u'' v''}\right] - \frac{1}{r}\frac{\partial}{\partial r}\left[r\,\overline{\rho'' v''^2}\right] + \left[\frac{\overline{\rho'' w''^2}}{r}\right] \tag{13.11}$$

It is advantageous to express the momentum transport quantities which appear on the right side of this equation in terms of the so-called Reynolds stresses which are here defined as follows.

$$\begin{aligned}
\tau_{xx} &= \frac{2}{3}\rho E - (\overline{\rho'' u''^2} - \rho u^2) \\
\tau_{rr} &= \frac{2}{3}\rho E - (\overline{\rho'' v''^2} - \rho v^2) \\
\tau_{\theta\theta} &= \frac{2}{3}\rho E - (\overline{\rho w''^2} - 0) \\
\tau_{xr} &= \tau_{rx} = 0 - (\overline{\rho'' u'' v''} - \rho u v) \\
\tau_{r\theta} &= \tau_{\theta r} = 0 \\
\tau_{\theta x} &= \tau_{x\theta} = 0
\end{aligned} \tag{13.12}$$

Notice that two of the six possible Reynolds stresses vanish in this case by reason of symmetry.

We also impose the restriction that the Reynolds stresses, as here defined, shall be purely deviatoric in character so that

$$\tau_{xx} + \tau_{rr} + \tau_{\theta\theta} = 0 \tag{13.13}$$

Upon adding the first three of Eqs. (13.12), then imposing the restriction expressed by Eq. (13.13) and rearranging the result slightly, we obtain

$$\rho E = \frac{1}{2}\left[(\overline{\rho'' u''^2} - \rho u^2) + (\overline{\rho'' v''^2} - \rho v^2) + (\overline{\rho'' w''^2} - 0) \right] \tag{13.14}$$

This shows clearly that ρE represents the mean turbulent energy per unit volume at any given space-time point. Also the quantity $\frac{2}{3}\rho E$ represents a pressure-like or isotropic term in Eqs. (13.12)

Upon eliminating the quantities having overbars between Eqs. (13.11) and (13.12) and regrouping terms, we obtain

$$\frac{\partial}{\partial t}[\rho u + \overline{\rho' u'}]$$

$$= -\frac{\partial}{\partial x}[p + \frac{2}{3}\rho E] - \frac{\partial}{\partial x}[\rho u^2 - \tau_{xx}] - \frac{1}{r}\frac{\partial}{\partial r}[r\,(\rho uv - \tau_{xr})] \tag{13.15}$$

$$\frac{\partial}{\partial t}[\rho v + \overline{\rho' v'}]$$

$$= -\frac{\partial}{\partial r}[p + \frac{2}{3}\rho E] - \frac{\partial}{\partial r}[\rho uv - \tau_{xr}] - \frac{1}{r}\frac{\partial}{\partial r}[r\,(\rho v^2 - \tau_{rr})] - [\frac{\tau_{\theta\theta}}{r}]$$

It is now possible to identify certain terms on the right side of Eqs. (13.15) as representing the components of the net resultant force per unit volume exerted upon the fluid element by the pressure and by the various Reynolds stresses. In this connection it is convenient to define an effective pressure term of the form

$$P = (p + \frac{2}{3}\rho E) \tag{13.16}$$

The resultant force components per unit volume produced by the purely deviatoric Reynolds stresses are now seen to be

$$\rho f_x = \left(\frac{\partial \tau_{xx}}{\partial x}\right) + \frac{1}{r}\frac{\partial}{\partial r}(r\ \tau_{xr})$$
$$\rho f_r = \left(\frac{\partial \tau_{xr}}{\partial x}\right) + \frac{1}{r}\frac{\partial}{\partial r}(r\ \tau_{rr}) - \frac{\tau_{\theta\theta}}{r} \tag{13.17}$$

It is also convenient at this point to express the turbulent mass transport effects in terms of a transport coefficient, that is,

$$\overline{\rho' u'} = -\ \kappa\varepsilon \left(\frac{\partial \rho}{\partial x}\right)$$
$$\overline{\rho' v'} = -\ \kappa\varepsilon \left(\frac{\partial \rho}{\partial r}\right) \tag{13.18}$$

By utilizing the notation of Eqs. (13.17) and (13.18), we can rewrite the momentum relations of Eqs. (13.15) in the following form

$$\frac{\partial}{\partial t}\left[\rho u - \kappa\varepsilon\left(\frac{\partial \rho}{\partial x}\right)\right] = -\frac{\partial}{\partial x}(\rho u^2) - \frac{1}{r}\frac{\partial}{\partial r}(r\ \rho uv) - \left(\frac{\partial P}{\partial x}\right) + \rho f_x$$
$$\frac{\partial}{\partial t}\left[\rho v - \kappa\varepsilon\left(\frac{\partial \rho}{\partial r}\right)\right] = -\frac{\partial}{\partial x}(\rho uv) - \frac{1}{r}\frac{\partial}{\partial r}(r\ \rho v^2) - \left(\frac{\partial P}{\partial r}\right) + \rho f_r \tag{13.19}$$

Once the Reynolds stresses and the resulting Reynolds forces be expressed in terms of a suitable eddy viscosity hypothesis, Eqs. (13.19) define the two basic momentum transport equations of our mathematical model.

Our next objective is to show how Eqs. (13.19) may be transformed into an alternative form which lends itself to a useful simplification. For this purpose we first expand these equations as follows

$$\rho\left(\frac{\partial u}{\partial t}\right) + u\left(\frac{\partial \rho}{\partial t}\right) - \frac{\partial}{\partial t}\left(\kappa\varepsilon\frac{\partial \rho}{\partial x}\right)$$

$$= - u\frac{\partial}{\partial x}(\rho u) - \rho u\left(\frac{\partial u}{\partial x}\right) - \frac{u}{r}\frac{\partial}{\partial r}(r\ \rho v) - \rho v\left(\frac{\partial u}{\partial r}\right) - \left(\frac{\partial P}{\partial x}\right) + \rho f_x$$

$$(13.20)$$

$$\rho\left(\frac{\partial v}{\partial t}\right) + v\left(\frac{\partial \rho}{\partial t}\right) - \frac{\partial}{\partial t}\left(\kappa\varepsilon\frac{\partial \rho}{\partial r}\right)$$

$$= - v\frac{\partial}{\partial x}(\rho u) - \rho u\left(\frac{\partial v}{\partial x}\right) - \frac{v}{r}\frac{\partial}{\partial r}(r\ \rho v) - \rho v\left(\frac{\partial v}{\partial r}\right) - \left(\frac{\partial P}{\partial r}\right) + \rho f_r$$

We now require the continuity relation, Eq. (12.5). In this connection it is convenient to define the following auxiliary variable, namely,

$$\rho G = \frac{\partial}{\partial x}\left(\kappa\varepsilon\frac{\partial \rho}{\partial x}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\ \kappa\varepsilon\frac{\partial \rho}{\partial r}\right) \qquad (13.21)$$

With this notation the continuity relation may be written as follows

$$-\left(\frac{\partial \rho}{\partial t}\right) = \frac{\partial}{\partial x}(\rho u) + \frac{1}{r}\frac{\partial}{\partial r}(r\ \rho v) + \rho G \qquad (13.22)$$

We next multiply Eq. (13.22) by u and add the result to the first of Eqs. (13.20). We also multiply Eq. (13.22) by v and add the result to the second of Eqs. (13.20). In this process certain terms cancel out. We then divide through by ρ. Finally the following results are obtained

$$\left(\frac{\partial u}{\partial t}\right) - \frac{1}{\rho}\frac{\partial}{\partial t}\left(\kappa\varepsilon\frac{\partial \rho}{\partial x}\right) = - u\left(\frac{\partial u}{\partial x}\right) - v\left(\frac{\partial u}{\partial r}\right) - \frac{1}{\rho}\left(\frac{\partial P}{\partial x}\right) + f_x - uG$$

$$(13.23)$$

$$\left(\frac{\partial v}{\partial t}\right) - \frac{1}{\rho}\frac{\partial}{\partial t}\left(\kappa\varepsilon\frac{\partial \rho}{\partial r}\right) = - u\left(\frac{\partial v}{\partial x}\right) - v\left(\frac{\partial v}{\partial r}\right) - \frac{1}{\rho}\left(\frac{\partial P}{\partial r}\right) + f_r - vG$$

Eqs. (13.23) are now exactly equivalent to Eqs. (13.19). These two pairs of equations differ only in form, not in essential content. However, Eqs. (13.23) lend themselves to a further useful simplification based on the assumption that

terms containing the factor $\kappa\varepsilon$ in Eqs. (13.23) and (13.21) may be neglected in comparison with the other terms involved. While this assumption seems plausible enough, its actual validity still remains to be verified quantitatively. Nevertheless, proceeding tentatively on this basis, we finally obtain the following simplified approximations.

$$\left(\frac{\partial u}{\partial t}\right) \doteq -u\left(\frac{\partial u}{\partial x}\right) - v\left(\frac{\partial u}{\partial r}\right) - \frac{1}{\rho}\left(\frac{\partial P}{\partial x}\right) + f_x$$
$$\left(\frac{\partial v}{\partial t}\right) \doteq -v\left(\frac{\partial v}{\partial x}\right) - v\left(\frac{\partial v}{\partial r}\right) - \frac{1}{\rho}\left(\frac{\partial P}{\partial r}\right) + f_r \qquad (13.24)$$

Thus we now have the option of incorporating into our mathematical model the momentum equations either in the more accurate form of Eqs. (13.19) or in the analytically simpler form of Eqs. (13.24). For the sake of definiteness, we state here that the subsequent analysis of this report is based specifically upon the simplified approximation shown in Eqs. (13.24). It appears, however, that if subsequent work shows it to be preferable to revert to the more accurate version, this can probably be accomplished without too much difficulty.

Notice that Eqs. (13.24) are exactly analogous to the equations of motion for laminar flow. However, the forces f_x and f_r, instead of being caused by deviatoric viscous stresses, are here caused by the deviatoric Reynolds stresses. The terms that have been dropped from Eqs. (13.24) are associated with mass diffusion effects and amount to a small correction to the effective acceleration of the fluid element.

14. Eddy Viscosity Hypothesis

It is instructive to expand the expressions for the various Reynolds stresses in the following manner. Consider first the mean turbulent kinetic energy as defined by Eq. (13.14). Let us begin by expanding the first term, namely,

$$\overline{(\rho''u''^2} - \rho u^2) = \overline{(\rho + \rho')\,(u + u')^2} - \rho u^2$$

$$= (\overline{\rho'u'^2} + 2u\,\overline{\rho'u'} + \cancel{\overline{\rho'}}^{\,0}\,u^2) + (\rho\overline{u'^2} + 2\rho u\,\cancel{\overline{u'}}^{\,0} + \cancel{\rho u^2}) - \cancel{\rho u^2}$$

$$= \overline{\rho'u'^2} + \rho\overline{u'^2} + 2u\,\overline{\rho'u'} \qquad (14.1)$$

Proceeding in like manner with the other two terms in Eq. (13.14), then adding and rearranging, we readily obtain the expression

$$\rho E = \frac{1}{2}\,(\overline{\rho'u'^2} + \overline{\rho'v'^2} + \overline{\rho'w'^2})$$

$$+ \frac{\rho}{2}\,(\overline{u'^2} + \overline{v'^2} + \overline{w'^2})$$

$$+ \quad (u\overline{\rho'u'} + v\,\overline{\rho'v'} + 0) \qquad (14.2)$$

Proceeding in like manner with Eqs. (13.12), and making use of Eq. (14.2) above, we obtain the following expressions for the deviatoric Reynolds stresses.

$$\tau_{xx} = -\left[\overline{\rho'u'^2} - \frac{1}{3}\,(\overline{\rho'u'^2} + \overline{\rho'v'^2} + \overline{\rho'w'^2})\right]$$

$$-\rho\left[\overline{u'^2} - \frac{1}{3}\,(\overline{u'^2} + \overline{v'^2} + \overline{w'^2})\right]$$

$$- \left[2u\,\overline{\rho'u'} - \frac{2}{3}\,(u\,\overline{\rho'u'} + v\,\overline{\rho'v'} + 0\right] \qquad (14.3)$$

$$\tau_{rr} = -\left[\overline{\rho' v'^2} - \frac{1}{3}\left(\overline{\rho' u'^2} + \overline{\rho' v'^2} + \overline{\rho' w'^2}\right)\right]$$

$$- \rho\left[\overline{v'^2} - \frac{1}{3}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)\right] \tag{14.3}$$

$$- \left[2v\,\overline{\rho' v'} - \frac{2}{3}\left(u\overline{\rho' u'} + v\overline{\rho' v'} + 0\right)\right]$$

$$\tau_{\theta\theta} = -\left[\overline{\rho' w'^2} - \frac{1}{3}\left(\overline{\rho' u'^2} + \overline{\rho' v'^2} + \overline{\rho' w'^2}\right)\right]$$

$$- \rho\left[\overline{w'^2} - \frac{1}{3}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)\right]$$

$$- \left[0 - \frac{2}{3}\left(u\overline{\rho' u'} + v\overline{\rho' v'} + 0\right]\right.$$

$$\tau_{xr} = \tau_{rx} = -\overline{\rho' u' v'} - \overline{\rho u' v'} - \left(v\overline{\rho' u'} + u\,\overline{\rho' v'}\right)$$

$$\tau_{r\theta} = \tau_{\theta r} = 0$$

$$\tau_{\theta x} = \tau_{x\theta} = 0$$

Notice that each of the Reynolds stresses in Eqs. (14.3) consists of three distinct terms. The first involves the transport of mass velocity, the second involves the transport of velocity, the third involves the transport of mass.

The mass flux effects may as usual be modelled heuristically in the form

$$\overline{\rho' u'} = -\kappa\varepsilon\left(\frac{\partial\rho}{\partial x}\right)$$

$$\overline{\rho' v'} = -\kappa\varepsilon\left(\frac{\partial\rho}{\partial r}\right) \tag{14.4}$$

The other terms are modelled heuristically according to the general reasoning explained earlier in section 4.

Referring back to Eq. (4.14), note that it is written in cartesian tensor notation. Upon translating it into cylindrical coordinates, and upon making use of Eqs. (4.15) and (4.16), we obtain the following results.

$$\tau_{xx} = \varepsilon \left\{ \left[\frac{\partial}{\partial x} (\rho u) - \frac{1}{3} \nabla \cdot (\rho \vec{V}) \right] + \rho \left[\left(\frac{\partial u}{\partial x}\right) - \frac{1}{3} \nabla \cdot \vec{V} \right] + \frac{2\kappa}{3} \left[2u \left(\frac{\partial \rho}{\partial x}\right) - v \left(\frac{\partial \rho}{\partial r}\right) \right] \right\}$$

$$\tau_{rr} = \varepsilon \left\{ \left[\frac{\partial}{\partial r} (\rho u) - \frac{1}{3} \nabla \cdot (\rho \vec{V}) \right] + \rho \left[\left(\frac{\partial v}{\partial r}\right) - \frac{1}{3} \nabla \cdot \vec{V} \right] + \frac{2\kappa}{3} \left[- u \left(\frac{\partial \rho}{\partial x}\right) + 2v \left(\frac{\partial \rho}{\partial r}\right) \right] \right\}$$

$$\tau_{\theta\theta} = \varepsilon \left\{ \left[\left(\frac{\rho v}{r}\right) - \frac{1}{3} \nabla \cdot (\rho \vec{V}) \right] + \rho \left[\left(\frac{v}{r}\right) - \frac{1}{3} \nabla \cdot \vec{V} \right] + \frac{2\kappa}{3} \left[- u \left(\frac{\partial \rho}{\partial x}\right) - v \left(\frac{\partial \rho}{\partial r}\right) \right] \right\}$$

$$\tau_{xr} = \tau_{rx} = \varepsilon \left\{ \frac{1}{2} \left[\frac{\partial}{\partial x} (\rho v) + \frac{\partial}{\partial r} (\rho u) \right] + \frac{\rho}{2} \left[\left(\frac{\partial v}{\partial x}\right) + \left(\frac{\partial u}{\partial r}\right) \right] + \kappa \left[v \left(\frac{\partial \rho}{\partial x}\right) + u \left(\frac{\partial \rho}{\partial r}\right) \right] \right\}$$

$$\tau_{r\theta} = \tau_{\theta r} = 0 \qquad (14.5)$$

$$\tau_{\theta x} = \tau_{x\theta} = 0$$

To show how these relations may be further simplified, consider the following expression which occurs in the first of Eqs. (14.5). Thus

$$\left[\frac{\partial}{\partial x} (\rho u) - \frac{1}{3} \nabla \cdot (\rho \vec{V}) \right] = \rho \left[\left(\frac{\partial u}{\partial x}\right) - \frac{1}{3} \nabla \cdot \vec{V} \right] + \left[u \left(\frac{\partial \rho}{\partial x}\right) - \frac{1}{3} \vec{V} \cdot \nabla \rho \right]$$

$$= \rho \left[\left(\frac{\partial u}{\partial x}\right) - \frac{1}{3} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{v}{r} \right) \right]$$

$$+ \left[u \left(\frac{\partial \rho}{\partial x}\right) - \frac{1}{3} \left(u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial r} \right) \right] \qquad (14.6)$$

$$= \frac{\rho}{3} \left[2 \left(\frac{\partial u}{\partial x}\right) - \left(\frac{\partial v}{\partial r} + \frac{v}{r} \right) \right] + \frac{1}{3} \left[2u \left(\frac{\partial \rho}{\partial x}\right) - v \left(\frac{\partial \rho}{\partial r}\right) \right]$$

With the aid of Eq. (14.6), the first of Eqs. (14.5) may readily be reduced to the form

$$\tau_{xx} = \varepsilon \left\{ \frac{2\rho}{3} \left[2 \left(\frac{\partial u}{\partial x}\right) - \left(\frac{\partial v}{\partial r} + \frac{v}{r}\right) \right] + \left(\frac{2K+1}{3}\right) \left[2u \left(\frac{\partial \rho}{\partial x}\right) - v \left(\frac{\partial \rho}{\partial r}\right) \right] \right\} \tag{16.7}$$

Proceeding in like manner with the other equations, we finally obtain the required results in the following form. These are equivalent to Eqs. (4.17)

$$\tau_{xx} = \varepsilon \left\{ \frac{2\rho}{3} \left[2 \left(\frac{\partial u}{\partial x}\right) - \left(\frac{\partial v}{\partial r} + \frac{v}{r}\right) \right] + \left(\frac{2K+1}{3}\right) \left[2u \left(\frac{\partial \rho}{\partial x}\right) + - v \left(\frac{\partial \rho}{\partial r}\right) \right] \right\}$$

$$\tau_{rr} = \varepsilon \left\{ \frac{2\rho}{3} \left[2 \left(\frac{\partial v}{\partial r}\right) - \left(\frac{v}{r} + \frac{\partial u}{\partial x}\right) \right] + \left(\frac{2K+1}{3}\right) \left[- u \left(\frac{\partial \rho}{\partial x}\right) + 2v \left(\frac{\partial \rho}{\partial r}\right) \right] \right\}$$

$$\tau_{\theta\theta} = \varepsilon \left\{ \frac{2\rho}{3} \left[2 \left(\frac{v}{r}\right) - \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial r}\right) \right] + \left(\frac{2K+1}{3}\right) \left[- u \left(\frac{\partial \rho}{\partial x}\right) - v \left(\frac{\partial \rho}{\partial r}\right) \right] \right\}$$

$$\tau_{xr} = \tau_{rx} = \varepsilon \left\{ \rho \left[\left(\frac{\partial v}{\partial x}\right) + \left(\frac{\partial u}{\partial r}\right) \right] + \left(\frac{2K+1}{2}\right) \left[v \left(\frac{\partial \rho}{\partial x}\right) + u \left(\frac{\partial \rho}{\partial r}\right) \right] \right\}$$

$$\tau_{r\theta} = \tau_{\theta r} = 0 \tag{14.8}$$

$$\tau_{\theta x} = \tau_{x\theta} = 0$$

Eqs. (14.8) represent the final eddy viscosity hypothesis as used in the present model. They relate the deviatoric Reynolds stresses to appropriate deviatoric components of the mean strain rate and density gradient. The quantity ε is treated as a true scalar, that is, a quantity whose magnitude is independent of the orientation in space of the local reference axes. In general the value of ε may vary from point to point, but in the present application ε may be taken as constant, at least to a first approximation. The experimental data that supports this simplifying assumption may be found, for example, in

the texts by Abramovich [1] and Schlichting [2].

It should not be overlooked that Eqs. (14.8), like Eqs. (14.4), are inherently approximate in character. This is true for every closure hypothesis.

15. Energy Equation

In applying the first law of thermodynamics, it is again convenient to deal with an infinitesimal control volume fixed in space. If e is the ordinary mean static internal energy per unit mass of the fluid, the mean total energy Q per unit volume may be defined as follows.

$$Q = \rho \left(e + \frac{u^2 + v^2}{2} + E\right) \tag{15.1}$$

The various components of the work done and of the energy transported by the fluid as it crosses the boundaries of the control volume make corresponding contributions to the local time rate of change of the total energy within the element. The aim of this section is to identify and evaluate each of these components and to assemble them into an overall energy equation.

According to the previous momentum analysis, the surface of the control volume is subject to an effective hydrostatic pressure of amount

$$P = p + \frac{2}{3} \rho E \tag{15.2}$$

where p is the ordinary mean fluid pressure and where $\frac{2}{3} \rho E$ represents the extra effect of the turbulent velocity fluctuations.

The effective pressure P does flow work on the system and contributes toward the local rate of change of total internal energy as follows.

$$\left(\frac{\partial Q}{\partial t}\right)_1 = - P\nabla\cdot\vec{V} - \vec{V}\cdot\nabla P = - \nabla\cdot (P\vec{V}) \tag{15.3}$$

Notice that the net flow work done by the effective pressure P may be expressed in two equivalent ways. The one term expression on the extreme right of Eq. (15.3) can readily be interpreted in relation to a control volume which remains fixed in space and whose boundaries are crossed by the mean flow. The other expression involving two terms can more easily be interpreted in relation to a control volume which moves with the mean flow. From the latter viewpoint, the term $-P\nabla\cdot\vec{V}$ denotes the work done by the effective pressure P on the rate of change of volume $\nabla\cdot\vec{V}$ of this moving element. Likewise the term $-\vec{V}\cdot\nabla P$ represents the work done by the resultant pressure force $-\nabla P$ on the rate of displacement $\vec{V}$ of the moving element. Eq. (15.3) involves a vector identity. This proves that the two interpretations offered above are entirely equivalent.

It should not be overlooked that Eq. (15.3) includes not only the flow work of the ordinary fluid pressure p, but also that which is produced by the turbulence effect $\frac{2}{3}\rho E$.

Next consider the transport of total energy Q across the boundaries of the element by the mean flow. This may be written

$$\left(\frac{\partial Q}{\partial t}\right)_2 = -\nabla\cdot(Q\vec{V}) \tag{15.4}$$

Of course internal energy is also transported by the turbulent fluctuations. We can model the turbulent flux in the usual way as follows

$$\overline{Q'\vec{V}'} = -\kappa\varepsilon\nabla Q \tag{15.5}$$

Therefore the corresponding contribution to the rate of change of internal energy is

$$\left(\frac{\partial Q}{\partial t}\right)_3 = -\nabla\cdot\left(\overline{Q'\vec{V}'}\right) = +\nabla\cdot(\kappa\varepsilon\nabla Q) \tag{15.6}$$

Before proceeding with the analysis of the remaining work effects, it is instructive to consider the sum of the three effects represented by Eqs. (15.3) (15.4) and (15.6). Thus

$$\left(\frac{\partial Q}{\partial t}\right)_1 + \left(\frac{\partial Q}{\partial t}\right)_2 + \left(\frac{\partial Q}{\partial t}\right)_3 = -\nabla\cdot(P\vec{V}) - \nabla\cdot(Q\vec{V}) + \nabla\cdot(\kappa\varepsilon\nabla Q)$$

$$= -\nabla\cdot\left[(P + Q)\,\vec{V} - \kappa\varepsilon\nabla Q\right]$$

$$= -\nabla\cdot(H\vec{V} - \kappa\varepsilon\nabla Q) \qquad (15.7)$$

where it has been convenient to define the auxiliary variable

$$H = P + Q \qquad (15.8)$$

The quantity H will be recognized as the mean effective total enthalpy per unit volume.

Finally, consider the net work done on the element by the deviatoric Reynolds stresses. This can be summarized in the form

$$\left(\frac{\partial Q}{\partial t}\right)_4 = \dot{W} + \rho\,\vec{f}\cdot\vec{V} \qquad (15.9)$$

In this expression $\dot{W}$ denotes the work done by the deviatoric stresses on the deviatoric strain rates of the mean flow. We may write this as

$$\dot{W} = \tau_{xx}\,\dot{\gamma}_{xx} + \tau_{rr}\,\dot{\gamma}_{rr} + \tau_{\theta\theta}\,\dot{\gamma}_{\theta\theta} + \tau_{xr}\,\dot{\gamma}_{xr} \qquad (15.10)$$

These stresses are as defined earlier in Eqs. (14.8). The corresponding deviatoric strain rates work out to be

$$\begin{aligned}
\dot{\gamma}_{xx} &= \frac{2}{3}\left[2\left(\frac{\partial u}{\partial x}\right) - \left(\frac{\partial v}{\partial r} + \frac{v}{r}\right)\right]\\
\dot{\gamma}_{rr} &= \frac{2}{3}\left[2\left(\frac{\partial v}{\partial r}\right) - \left(\frac{v}{r} + \frac{\partial u}{\partial x}\right)\right]\\
\dot{\gamma}_{\theta\theta} &= \frac{2}{3}\left[2\left(\frac{v}{r}\right) - \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial r}\right)\right]\\
\dot{\gamma}_{xr} &= \left[\left(\frac{\partial v}{\partial x}\right) + \left(\frac{\partial u}{\partial r}\right)\right]
\end{aligned} \qquad (15.11)$$

Now referring back to the last term of Eq. (15.9), we recognize that $\rho\vec{f}$ denotes the net force per unit volume exerted upon the element by the purely deviatoric Reynolds stresses. The components ρf_x and ρf_r of this force were previously defined in Eqs. (13.18). Of course the corresponding resultant force exerted by the effective pressure P, namely,

$$-\nabla P = -\nabla\left(p + \frac{2}{3}\rho E\right) \tag{15.12}$$

has already been included in Eq. (15.3) and hence is not involved in Eq. (15.9).

The desired overall energy equation can now be obtained by adding Eqs. (15.7) and (15.9). The result is

$$\left(\frac{\partial Q}{\partial t}\right)_1 + \left(\frac{\partial Q}{\partial t}\right)_2 + \left(\frac{\partial Q}{\partial t}\right)_3 + \left(\frac{\partial Q}{\partial t}\right)_4 =$$

$$\left(\frac{\partial Q}{\partial t}\right) = \dot{Q} = -\nabla\cdot\left(H\vec{V} - \kappa\varepsilon\nabla Q\right) + \dot{W} + \rho\vec{f}\cdot\vec{V} \tag{15.13}$$

This can now be translated into cylindrical coordinates as follows.

$$\dot{Q} = -\frac{\partial}{\partial x}\left[Hu - \kappa\varepsilon\left(\frac{\partial Q}{\partial x}\right)\right] - \frac{1}{r}\frac{\partial}{\partial r}\left\{r\left[Hv - \kappa\varepsilon\left(\frac{\partial Q}{\partial r}\right)\right]\right\}$$
$$+ \dot{W} + \rho f_x u + \rho f_r v \tag{15.14}$$

In earlier sections of this report it has been shown how the continuity equation fixes the quantity $\left(\frac{\partial \rho}{\partial t}\right)$ and how the momentum equations fix the quantities $\left(\frac{\partial u}{\partial t}\right)$ and $\left(\frac{\partial v}{\partial t}\right)$. We now wish to show how the addition of the above energy relation, Eq. (15.14), to our mathematical model fixes the quantity $\left(\frac{\partial p}{\partial t}\right)$.

In this connection recall that for a perfect gas with constant specific heats

$$\rho e = \left(\frac{p}{RT}\right)\left(\frac{R}{\gamma-1}T\right) = \left(\frac{1}{\gamma-1}\right)p \tag{15.15}$$

Hence Eq. (15.1) can now be rewritten as

$$Q = \left(\frac{1}{\gamma-1}\right) p + \rho \left(\frac{u^2 + v^2}{2} + E\right) \tag{15.16}$$

Upon differentiating this equation with respect to time and solving for $\left(\frac{\partial p}{\partial t}\right)$ we readily obtain

$$\left(\frac{\partial p}{\partial t}\right) = (\gamma-1) \left\{\dot{Q} - \left(\frac{u^2 + v^2}{2} + E\right) \left(\frac{\partial \rho}{\partial t}\right) - \rho \left[u \left(\frac{\partial u}{\partial t}\right) + v \left(\frac{\partial v}{\partial t}\right) + \left(\frac{\partial E}{\partial t}\right)\right]\right\} \tag{15.17}$$

This is the result required. Note that $\dot{Q}$ is fixed by the energy relation, Eq. (15.14) while $\left(\frac{\partial \rho}{\partial t}\right)$, $\left(\frac{\partial u}{\partial t}\right)$ and $\left(\frac{\partial v}{\partial t}\right)$ are fixed by the continuity and momentum equations as already noted. Since E is here treated as a known function, the quantity $\left(\frac{\partial E}{\partial t}\right)$ is likewise known.

The development to this point has shown how the continuity, momentum and energy equations serve to fix the quantities $\left(\frac{\partial \rho}{\partial t}\right)$, $\left(\frac{\partial u}{\partial t}\right)$, $\left(\frac{\partial v}{\partial t}\right)$ and $\left(\frac{\partial p}{\partial t}\right)$, assuming that ε and E are known functions and assuming that the applicable boundary conditions are properly specified. If we hold these boundary conditions constant and integrate numerically through time, the solution should ultimately converge to the corresponding steady state.

There is an aspect of the turbulent flux of energy Q as represented in Eq. (15.5) that is relevant to the analysis of the second law of thermodynamics as developed in the next section. To explain this, we expand Eq. (15.5) in the following way

$$\begin{aligned}
\overline{Q'\vec{V}'} &= - \kappa\varepsilon\nabla Q = - \kappa\varepsilon\nabla \left[\rho \left(e + \frac{u^2 + v^2}{2} + E\right)\right] \\
&= - \kappa\varepsilon\nabla \left[\rho \left(\frac{RT}{\gamma-1} + \frac{u^2 + v^2}{2} + E\right)\right] \\
&= - \kappa\varepsilon \left\{\frac{R}{(\gamma-1)} [\rho\nabla T + T\nabla\rho] \right. \\
&\qquad \left. + \nabla \left[\rho \left(\frac{u^2 + v^2}{2} + E\right)\right]\right\}
\end{aligned} \tag{15.18}$$

Of the three terms on the right side of Eq. (15.18), only the first term involves the temperature gradient ∇T . Recall that in steady laminar flow, the net heat flux associated with molecular conduction is expressed in the form of the Fourier equation

$$\vec{q} = - k\nabla T \tag{15.19}$$

where k is the ordinary molecular thermal conductivity of the fluid.

By analogy with this, we see that the first term of Eq. (15.18), since it involves the local temperature gradient ∇T , may be interpreted as an apparent or equivalent heat flux which is produced by the turbulent mixing. Thus

$$\vec{q} = -\left[\kappa\varepsilon \left(\frac{R}{\gamma-1}\right)\cdot \rho\right] \nabla T \qquad (15.20$$

Comparison of these last two expressions shows that the effective thermal conductivity, which is now associated not with the molecular action but with the turbulent mixing, is simply

$$k = \kappa\varepsilon \left(\frac{R}{\gamma-1}\right) \rho \tag{15.21}$$

Perhaps it should also be remarked in passing that, assuming that the independent properties p and ρ are specified over the field, the temperature T which figures in the above relations may of course be found from the equation of state of a perfect gas. We may write this here in the form

$$T = \frac{p}{R\rho} \tag{15.23}$$

16. Second Law of Thermodynamics

The extension of the analysis of the preceding section to the second law is straightforward. If p and ρ are the independent thermodynamic properties, then for a perfect gas with constant specific heats the entropy per unit mass may be written

$$s = \left(\frac{R}{\gamma-1}\right)\left[\ell n \left(\frac{p}{p_o}\right) - \gamma \ \ell n \left(\frac{\rho}{\rho_o}\right)\right] \tag{16.1}$$

where p_o , ρ_o denote some convenient reference state at which the entropy is taken as zero, by definition. Inlet stagnation conditions provide such a convenient reference state.

We define the corresponding entropy per unit volume simply as

$$S = \rho s \tag{16.2}$$

Now the work term $\dot{W}$, the apparent heat flux $\vec{q}$, and the mean and turbulent transport of entropy contribute to the overall time rate of change of entropy per unit volume. For example, the transport of total entropy by the mean flow and by the turbulent fluctuations gives

$$\left(\frac{\partial S}{\partial t}\right)_1 = - \nabla \cdot \ [S\vec{V} - \kappa\varepsilon\nabla S] \tag{16.3}$$

Of the various work terms considered in the previous section, only the quantity $\dot{W}$ affects the entropy. The relation is simply

$$\left(\frac{\partial S}{\partial t}\right)_2 = \frac{\dot{W}}{T} \tag{16.4}$$

It has been shown that a portion of the enthalpy transport amounts to an apparent heat flux $\vec{q}$. This contributes to the rate of change of entropy according to the relation

$$\left(\frac{\partial S}{\partial t}\right)_3 = - \nabla \cdot \left(\frac{\vec{q}}{T}\right) = + \nabla \cdot \left[\kappa\varepsilon \left(\frac{R}{\gamma-1}\right) \frac{\rho}{T} \nabla T\right] \tag{16.5}$$

The desired result is now found by adding these three effects. In this way we obtain the equation

$$\left(\frac{\partial S}{\partial t}\right)_1 + \left(\frac{\partial S}{\partial t}\right)_2 + \left(\frac{\partial S}{\partial t}\right)_3 = \left(\frac{\partial S}{\partial t}\right)$$

$$= \dot{S} = - \nabla \cdot \left[S\vec{V} - \kappa\varepsilon\nabla S - \kappa\varepsilon \left(\frac{R}{\gamma-1}\right) \frac{\rho}{T} \nabla T\right] + \frac{\dot{W}}{T} \tag{16.6}$$

Translating this into cylindrical coordinates in the usual way gives

$$\dot{S} = - \frac{\partial}{\partial x} \left[Su - \kappa\varepsilon \left(\frac{\partial S}{\partial x}\right) - \kappa\varepsilon \left(\frac{R}{\gamma-1}\right) \frac{\rho}{T} \left(\frac{\partial T}{\partial x}\right)\right]$$

$$- \frac{1}{r} \frac{\partial}{\partial r} \left\{r \left[Sv - \kappa\varepsilon \left(\frac{\partial S}{\partial r}\right) - \kappa\varepsilon \left(\frac{R}{\gamma-1}\right) \frac{\rho}{T} \left(\frac{\partial T}{\partial r}\right)\right]\right\} + \frac{\dot{W}}{T} \tag{16.7}$$

The quantity $\dot{S}$ defined by Eq. (16.7) can now be further developed in the following way. Firstly, we differentiate Eqs. (16.2) and (16.1) with respect to time. This gives

$$\dot{S} = \rho \left(\frac{\partial s}{\partial t}\right) + s \left(\frac{\partial \rho}{\partial t}\right) \tag{16.8}$$

and

$$\left(\frac{\partial s}{\partial t}\right) = \left(\frac{R}{\gamma-1}\right) \left[\frac{1}{p} \left(\frac{\partial p}{\partial t}\right) - \frac{\gamma}{\rho} \left(\frac{\partial \rho}{\partial t}\right)\right] \tag{16.9}$$

Secondly, we combine these last two expressions and rearrange the result. This gives finally

$$\left(\frac{\dot{S}}{R}\right) - \left(\frac{1}{\gamma-1}\right) \frac{\rho}{p} \left(\frac{\partial p}{\partial t}\right) + \left(\frac{\gamma}{\gamma-1} - \frac{s}{R}\right) \left(\frac{\partial \rho}{\partial t}\right) = z = 0 \tag{16.10}$$

where $\dot{S}$ is found from Eq. (16.7), $\left(\frac{\partial p}{\partial t}\right)$ is found from Eq. (15.16) and $\left(\frac{\partial \rho}{\partial t}\right)$ is found from Eq. (12.5).

The auxiliary variable z is introduced into Eq. (16.10) as a convenient abbreviation to denote the sum of all of the terms on the left side. Thus the requirements of the second law can now be stated concisely as $z = 0$.

The earlier discussion has shown that Eq. (16.10) is not in general satisfied at interior points of a computational cell if the boundary conditions along the exit cross-section of the cell be specified arbitrarily. These exit conditions must be so adjusted, by a suitable process of numerical relaxation, as to satisfy the condition $z = 0$, at all interior points of the cell. The method of accomplishing this has already been outline in section 10.

17. Further Development of Boundary Conditions

It was shown earlier that along a fixed wall normal to the axis, the four boundary conditions prescribed by Eqs. (9.1) must be satisfied. These are repeated below for convenience.

$$\begin{aligned} &u = 0 \\ &\left(\frac{\partial u}{\partial t}\right) = 0 \\ &\left(\frac{\partial T}{\partial x}\right) = 0 \\ &v = \pm 11.0 \sqrt{\frac{|\tau_{xr}|}{\rho}} \end{aligned} \tag{17.1}$$

We are now in possession of additional relations which allow a significant simplification of these boundary conditions. Thus from the first of the simplified equations of motion, Eqs. (13.24), we have

$$\left(\frac{\partial u}{\partial t}\right) = - u \left(\frac{\partial u}{\partial x}\right) - v \left(\frac{\partial u}{\partial r}\right) - \frac{1}{\rho} \left(\frac{\partial P}{\partial x}\right) + f_x \tag{17.2}$$

This now reduces to

$$0 = - 0 - 0 - \frac{1}{\rho}\frac{\partial}{\partial x} \left[p + \frac{2}{3}\rho E\right] + f_x \qquad (17.3)$$

From the third of Eqs. (17.1) we also have

$$R\left(\frac{\partial T}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{p}{\rho}\right) = \frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right) - \frac{p}{\rho^2}\left(\frac{\partial \rho}{\partial x}\right) = 0 \qquad (17.4)$$

By algebraic reduction of Eqs. (17.3) and (17.4) we readily obtain the interesting relations below.

$$\frac{1}{p}\left(\frac{\partial p}{\partial x}\right) = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial x}\right) = \left\{\frac{f_x - \frac{2}{3}\left(\frac{\partial E}{\partial x}\right)}{\frac{p}{\rho} + \frac{2}{3}E}\right\} \qquad (17.5)$$

Notice that f_x represents the deviatoric Reynolds force normal to the wall. While the component f_r tangential to the wall might be significant, the component normal to the wall is almost certain to be negligible.

Next consider the quantity $\left(\frac{\partial E}{\partial x}\right)$. Since E itself is normally rather small compared with p/ρ, this derivative is unlikely to be significant outside the viscous sublayer. Moreover, E normally passes through a maximum at or near the edge of the viscous sublayer and at this maximum the derivative vanishes. We conclude that $\left(\frac{\partial E}{\partial x}\right)$, like f_x itself, can safely be neglected.

On this basis the boundary conditions resulting from Eqs. (17.1) and (17.5) simplify to the form

$$
\begin{aligned}
u &= 0 \\
v &\doteq \pm\, 11.0\sqrt{\frac{|\tau_{xr}|}{\rho}} \\
\left(\frac{\partial p}{\partial x}\right) &\doteq 0 \\
\left(\frac{\partial \rho}{\partial x}\right) &\doteq 0
\end{aligned}
\tag{17.6}
$$

Similarly, the boundary conditions along the outer wall simplify to the form

$$
\begin{aligned}
u &\doteq \pm\, 11.0\sqrt{\frac{|\tau_{xr}|}{\rho}} \\
v &= 0 \\
\left(\frac{\partial p}{\partial r}\right) &\doteq 0 \\
\left(\frac{\partial \rho}{\partial r}\right) &\doteq 0
\end{aligned}
\tag{17.7}
$$

Recall from section 9 that in Eq. (17.6) the algebraic sign of v must agree with that of τ_{xr} while in Eq. (17.7) the algebraic sign of u must be opposite to that of τ_{xr}. These simple rules may readily be verified by examining the directions of the stress and velocity components acting on a fluid element near a wall.

It is well known that outside the viscous sublayer in fully developed pipe flow the axial velocity distribution is well described by a universal logarithmic law of the form

$$\left(\frac{u}{u^*}\right) = A\ \ell n\left(\frac{u^* y}{\nu}\right) + B \tag{17.8}$$

where u^* is the so called friction velocity. If $|\tau_{xr}|$ denotes the magnitude of the shear stress at the wall, the friction velocity is

$$u^* = \sqrt{\frac{|\tau_{xr}|}{\rho}} \qquad (17.9)$$

Numerous experiments have established the following values as good average estimates of constants A and B , namely,

$$A = 2.5 \qquad B = 5.0 \qquad (17.9)$$

In the very thin laminar sublayer immediately adjacent to a smooth wall, the velocity distribution is well described by the linear relation

$$\left(\frac{u}{u^*}\right) = \left(\frac{u^* y}{\nu}\right) = y^+ \qquad (17.10)$$

Between the laminar sublayer and the fully turbulent region, the velocity distribution gradually changes from that of Eq. (17.10) to that of Eq. (17.8). It is convenient, however, to ignore the details of this gradual transition and instead to plot the two curves as if they were independent. The solution is taken as following Eq. (17.10) up to the point of intersection of the two curves, and as following Eq. (17.8) beyond that point. This point of intersection is defined as the theoretical edge of the viscous sublayer.

At this location we must have

$$\left(\frac{u}{u^*}\right) = y^+ = 2.5 \ln y^+ + 5.0 \qquad (17.11)$$

Numerical trial and error verifies that this relation is satisfied for

$$\left(\frac{u}{u^*}\right) = y^+ = 10.99 \doteq 11.0$$

Hence it follows that

$$u \doteq \pm \; 11.0\sqrt{\frac{|\tau_{xr}|}{\rho}} \tag{17.12}$$

Of course, the accuracy of this analytical wall function is somewhat variable and dependent on various circumstances. It can be made more general by introducing further complications, but such complications are considered to be not justified in the present context.

Further information on the so-called law of the wall as expressed by Eq. (17.8) may be found in many standard texts including that of Schlichting [2]. Other discussions of wall conditions and wall functions are presented, for example, in references [68] through [75].

18. Summary of Principal Equations

All of the principal equations that characterize the present mathematical model of the flow are summarized in this section.

Part (a) summarizes the boundary conditions that must be satisfied at points which lie along various portions of the enclosing contour of the cell.

Parts (b), (c) and (d) summarize the principal equations in the approximate order in which they would be used to calculate the small change of state occurring in some small time interval Δt at a typical interior point of the cell. Part (b) lists the various auxiliary variables that occur at each point. Part (c) summarizes the four fundamental laws that yield the quantities $(\frac{\partial u}{\partial t})$, $(\frac{\partial v}{\partial t})$, $(\frac{\partial p}{\partial t})$, $(\frac{\partial \rho}{\partial t})$. These are the two momentum equations, the continuity equation and the energy equation. Part (d) summarizes the four simple relations that fix the resulting new state of the system at time $(t + \Delta t)$.

Of course the above calculations must be carried out for all points of the cell, both boundary and interior points, in accordance with the respective equations that apply to each.

The above sequence must be repeated over as many successive time intervals as necessary to reach an essentially steady state. During this calculation the boundary conditions around the cell, including the boundary conditions across the exit section, are held fixed.

The present discussion assumes that the eddy viscosity ε , the turbulent energy E and its time derivative $(\frac{\partial E}{\partial t})$ are known at every space-time point. Fortunately, it happens that the simple assumption that ε is constant over the field provides a reasonable first approximation and eliminates many complications. Moreover, it is also permissible in a first approximation simply to neglect E and $(\frac{\partial E}{\partial t})$. Nevertheless, these terms are retained in the following equations for the sake of greater generality.

For a more general analysis it might become desirable to model the quantities ε , E and $(\frac{\partial E}{\partial t})$ themselves by means of suitable additional equations. This aspect lies outside the scope of the present report. Nevertheless, however such possible additional complications are handled, the essential features of the present analysis are not significantly altered thereby.

The calculations of part (a) start with known or assumed values of the quantities ε , E and $\frac{\partial E}{\partial t}$, and of the independent variables u , v , p , ρ at time t .

It has been pointed out earlier that the solution obtained by the above process does not in general satisfy the second law of thermodynamics because the boundary conditions across the exit section of the cell are as yet

arbitrary. To determine the extent to which the above provisional solution deviates from the requirements of the second law, we must carry out the additional calculations summarized in part (e) of this section.

In order to bring the final solution into compliance with the second law we must relax the exit boundary conditions according to the general method described in section 10. This process is summarized in part (f) of the present section.

Periodically during the above relaxation process of parts (e) and (f), the calculation procedure summarized in parts (a), (b), (c) and (d) of this section must be repeated to ensure not only that the second law is satisfied, but also that the four original basic equations continue to be satisfied. Thus the calculations finally required to satisfy all five governing laws over all interior points of a single cell are bound to be very lengthy.

On the other hand once the solution over a given cell is finally found, it need never be revised further. In other words the calculations in any given cell never affect the solution in any other cell upstream of the given cell. This amounts to saying that the flow field, instead of being elliptical, is piecewise parabolic. Hence the solution can simply be marched downstream cell by cell as far as may be required. The computation, while still formidable, is nevertheless greatly simplified by the fact that it suffices to deal with but one cell at a time.

(a) Simplified Boundary Conditions

At Points Along Cross Section of Entering Jet

$$\left.\begin{array}{l} u \\ v \\ p \\ \rho \end{array}\right\} \quad \text{Known (Basic data of problem)} \tag{18.1}$$

At Points Along Wall Normal to Axis

$$u = 0$$

$$v = \pm\, 11.0 \; \frac{|\tau_{xr}|}{\rho} \qquad \text{Algebraic sign of } v \text{ must agree with that of } \tau_{xr}$$

$$\left(\frac{\partial p}{\partial x}\right) \doteq 0 \tag{18.2}$$

$$\left(\frac{\partial \rho}{\partial x}\right) \doteq 0$$

At Points Along Pipe Wall

$$u \doteq \pm\, 11.0 \; \frac{|\tau_{xr}|}{\rho} \qquad \text{Algebraic sign of } u \text{ must be opposite to that of } \tau_{xr}$$

$$v = 0 \tag{18.3}$$

$$\left(\frac{\partial p}{\partial r}\right) \doteq 0$$

$$\left(\frac{\partial \rho}{\partial r}\right) \doteq 0$$

At Points Along Pipe Axis

$$v = 0$$

$$\left.\begin{array}{l} u \\ p \\ \rho \end{array}\right\}$$ These values are fixed by the axial momentum equation, the continuity equation and the momentum equation, with $v = 0$ and $\frac{\partial}{\partial r} = 0$. (18.4)

At Points Along Exit Plane of Cell

$$\left.\begin{array}{l} u \\ v \\ p \\ \rho \end{array}\right\}$$ These values, which are initially guessed, must be gradually adjusted so that the final solution satisfies, in addition to the other four basic laws, the second law of thermodynamics at all interior points. The method of accomplishing this is summarized in parts (e) and (f) below. (18.5)

At Points Along Entrance Cross-Section of Any Cell Except the First

$$\left.\begin{matrix} u \\ v \\ p \\ \rho \end{matrix}\right\}$$ These values are identical with those across the exit section of the preceding cell. They are therefore known.

(b) Auxiliary Variables

$$P = p + \frac{2}{3}\rho E \tag{18.7}$$

$$Q = \frac{p}{(\gamma-1)} + \rho\left(\frac{u^2+v^2}{2} + E\right) \tag{18.8}$$

$$H = P + Q \tag{18.9}$$

$$\dot{\gamma}_{xx} = \frac{2}{3}\left[2\left(\frac{\partial u}{\partial x}\right) - \left(\frac{\partial v}{\partial r} + \frac{v}{r}\right)\right] \tag{18.10}$$

$$\dot{\gamma}_{rr} = \frac{2}{3}\left[2\left(\frac{\partial v}{\partial r}\right) - \left(\frac{v}{r} + \frac{\partial u}{\partial x}\right)\right] \tag{18.11}$$

$$\dot{\gamma}_{\theta\theta} = \frac{2}{3}\left[2\left(\frac{v}{r}\right) - \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial r}\right)\right] \tag{18.12}$$

$$\dot{\gamma}_{xr} = \left[\left(\frac{\partial v}{\partial x}\right) + \left(\frac{\partial u}{\partial r}\right)\right] \tag{18.13}$$

$$\tau_{xx} = \varepsilon\left\{\rho\,\dot{\gamma}_{xx} + \left(\frac{2\kappa+1}{3}\right)\left[2u\left(\frac{\partial \rho}{\partial x}\right) - v\left(\frac{\partial \rho}{\partial r}\right)\right]\right\} \tag{18.14}$$

$$\tau_{rr} = \varepsilon\left\{\rho\,\dot{\gamma}_{rr} + \left(\frac{2\kappa+1}{3}\right)\left[-u\left(\frac{\partial \rho}{\partial x}\right) + 2v\left(\frac{\partial \rho}{\partial r}\right)\right]\right\} \tag{18.15}$$

$$\tau_{\theta\theta} = \varepsilon\left\{\rho\,\dot{\gamma}_{\theta\theta} + \left(\frac{2\kappa+1}{3}\right)\left[-u\left(\frac{\partial \rho}{\partial x}\right) - v\left(\frac{\partial \rho}{\partial r}\right)\right]\right\} \tag{18.16}$$

$$\tau_{xr} = \varepsilon\left\{\rho\,\dot{\gamma}_{xr} + \left(\frac{2\kappa+1}{2}\right)\left[v\left(\frac{\partial \rho}{\partial x}\right) + u\left(\frac{\partial \rho}{\partial r}\right)\right]\right\} \tag{18.17}$$

$$\dot{W} = \tau_{xx}\,\dot{\gamma}_{xx} + \tau_{rr}\,\dot{\gamma}_{rr} + \tau_{\theta\theta}\,\dot{\gamma}_{\theta\theta} + \tau_{xr}\,\dot{\gamma}_{xr} \tag{18.18}$$

$$\rho f_x = \left(\frac{\partial \tau_{xx}}{\partial x}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\tau_{xr}\right) \tag{18.19}$$

$$\rho f_r = \left(\frac{\partial \tau_{xr}}{\partial x}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\tau_{rr}\right) - \frac{\tau_{\theta\theta}}{r} \tag{18.20}$$

$$\dot{Q} = -\frac{\partial}{\partial x}\left[Hu - \kappa\varepsilon\left(\frac{\partial Q}{\partial x}\right)\right] - \frac{1}{r}\frac{\partial}{\partial r}\left[r\left(Hv - \kappa\varepsilon\frac{\partial Q}{\partial r}\right)\right]$$

$$+ \dot{W} + \rho f_x u + \rho f_r v \tag{18.21}$$

(c) Time Rates of Change

$$\left(\frac{\partial u}{\partial t}\right) = - u \left(\frac{\partial u}{\partial x}\right) - v \left(\frac{\partial u}{\partial r}\right) - \frac{1}{\rho} \left(\frac{\partial P}{\partial x}\right) + f_x \tag{18.22}$$

$$\left(\frac{\partial v}{\partial t}\right) = - u \left(\frac{\partial v}{\partial x}\right) - v \left(\frac{\partial v}{\partial r}\right) - \frac{1}{\rho} \left(\frac{\partial P}{\partial r}\right) + f_r \tag{18.23}$$

$$\left(\frac{\partial \rho}{\partial t}\right) = - \frac{\partial}{\partial x} \left[\rho u - \kappa\varepsilon \left(\frac{\partial \rho}{\partial x}\right)\right] - \frac{\partial}{\partial r} \left[r\left(\rho v - \kappa\varepsilon \frac{\partial \rho}{\partial r}\right)\right] \tag{18.24}$$

$$\left(\frac{\partial p}{\partial t}\right) = (\gamma-1) \left\{\dot{Q} - \left(\frac{u^2 + v^2}{2}\right) \left(\frac{\partial \rho}{\partial t}\right) - \rho \left[u \left(\frac{\partial u}{\partial t}\right) + v \left(\frac{\partial v}{\partial t}\right) + \left(\frac{\partial E}{\partial t}\right)\right]\right\} \tag{18.25}$$

(d) New State of System

$$u_{(n+1)} = u_n + \left(\frac{\partial u}{\partial t}\right)_n \Delta t \tag{18.26}$$

$$v_{(n+1)} = v_n + \left(\frac{\partial v}{\partial t}\right)_n \Delta t \tag{18.27}$$

$$\rho_{n+1} = \rho_n + \left(\frac{\partial \rho}{\partial t}\right)_n \Delta t \tag{18.28}$$

$$p_{n+1} = p_n + \left(\frac{\partial p}{\partial t_n}\right) \Delta t \tag{18.29}$$

(e) Second Law

$$T = \frac{p}{RT} \tag{18.30}$$

$$s = \left(\frac{R}{\gamma-1}\right) \left[\ell n \left(\frac{p}{p_o}\right) - \gamma \ell n \left(\frac{\rho}{\rho_o}\right)\right] \tag{18.31}$$

$$S = \rho s \tag{18.32}$$

$$\begin{aligned} \dot{S} = &- \frac{\partial}{\partial x} \left[Su - \kappa\varepsilon \left(\frac{\partial S}{\partial x}\right) - \kappa\varepsilon \left(\frac{R}{\gamma-1}\right) \frac{\rho}{T} \left(\frac{\partial T}{\partial x}\right)\right] \\ &- \frac{1}{r} \left\{r \left[Sv - \kappa\varepsilon \left(\frac{\partial S}{\partial r}\right) - \kappa\varepsilon \left(\frac{R}{\gamma-1}\right) \frac{\rho}{T} \left(\frac{\partial T}{\partial r}\right)\right]\right\} + \frac{\dot{W}}{T} \end{aligned} \tag{18.33}$$

$$\left(\frac{\dot{S}}{R}\right) - \left(\frac{1}{\gamma-1}\right) \frac{\rho}{p} \left(\frac{\partial p}{\partial t}\right) + \left(\frac{\gamma}{\gamma-1} - \frac{s}{R}\right) \left(\frac{\partial \rho}{\partial t}\right) = z = 0 \tag{18.34}$$

(f) Relaxation of Cell Exit Conditions

Notation:

$m = 1,2,3, \ldots n$ = first index denoting radial position of mesh point in cell

$k = 1,2,3,4,5,6$ = second index denoting axial position of mesh point in cell

$N = 4\ (n-1)$ = total number of interior mesh points in cell = total number of initially undetermined boundary parameters across exit section of cell (18.35)

$i = 1,2,3, \ldots N$ = index denoting interior mesh point of cell

$j = 1,2,3, \ldots N$ = index denoting intially undetermined boundary parameter across exit of cell.

The exit boundary parameters are relabelled according to the following scheme.

$$
\begin{array}{llll}
u_{1,6} = \alpha_1 & u_{2,6} = \alpha_5 & ---- & u_{n,6} = \alpha_{(N-3)} \\
v_{1,6} = \alpha_2 & v_{2,6} = \alpha_6 & ---- & v_{n,6} = \alpha_{(N-2} \\
p_{1,6} = \alpha_3 & p_{2,6} = \alpha_7 & ---- & p_{n,6} = \alpha_{(N-1)} \\
\rho_{1,6} = \alpha_4 & \rho_{2,6} = \alpha_8 & ---- & \rho_{n,6} = \alpha_N
\end{array}
\qquad (18.36)
$$

An arbitrary small change $\delta\alpha_j$ in exit parameter α_j produces small changes in the second law discrepancy of amount

$$\delta z_i = \left(\frac{\partial z_i}{\partial \alpha j}\right) \delta\alpha j \qquad (18.37)$$

at each interior grid point $j = 1,2,3, \ldots N$. These changes can be found by calculation according to the preceding principles.

At any stage in the relaxation process, the optimum change in exit parameter αj is given by the relaxation formula

$$(\delta\alpha_j) = - \frac{\sum_{i=1}^{N} r_i \, z_i \, (\frac{\partial z_i}{\partial \alpha_j})}{\sum_{i=1}^{N} r_i \, (\frac{\partial z_i}{\partial \alpha_j})^2} \tag{18.38}$$

The above relaxation process is continued until the following criterion is satisfied

$$\sum_{i=1}^{N} r_i \, z_i^2 \leq I_o \tag{18.39}$$

where I_o is some very small preassigned quantity that defines the acceptable level of error in satisfying the second law.

19. Classification of Equations by Determinant Method

This discussion refers to the final equations of section 18. For the purposes of the present argument, it is permissible to treat ε , E and $(\frac{\partial E}{\partial t})$ as known functions of space and time.

The principal equations are now the following.

Momentum, x direction	Eq. (18.22)
Momentum, y direction	Eq. (18.23)
Energy	Eq. (18.25)
Continuity	Eq. (18.24)

The order of Eqs. (18.24) and (18.25) has been reversed in the above listing because this ultimately simplifies the format of the basic matrix that is involved in this analysis.

Our aim in this section is to establish rigorously whether the above set of equations is of elliptical, parabolic or hyperbolic type. This question must be settled in order to verify that the nature of the solution procedure and of the boundary conditions summarized in section 18 are in fact correct and appropriate to the problem.

In this connection we note that all quantities which occur in the above four basic equations can be expressed as functions of the four basic variables u , v , p , ρ and of their various partial derivatives. This can be shown by expanding Eqs. (18.7) through (18.21) and substituting the results into Eqs. (18.22) through (18.25). Details are shown in section 20.

The highest derivatives which occur in the four resulting equations are found to be u_{xx}, u_{xr}, u_{rr}, v_{xx}, v_{xr}, v_{rr}, p_{xx}, p_{xr}, p_{rr} ρ_{xx}, ρ_{xr}, ρ_{rr} . It is convenient for our present purpose to relabel these quantities x_1, x_2, x_3, ... x_{12} . Using this notation we can rewrite the above four basic equations in matrix format. Specifically, the above four equations are shown in rows 2, 5, 8 and 11 of Table 19.1. In rows $r = 2, 5, 8$ or 11 , all terms in quantities other than the above twelve x_i's are transferred to the right side of the equation. The detailed algebraic form of these auxiliary terms is not needed for our present purpose and may therefore be ignored.

In addition to rows 2, 5, 8 and 11, Table 19.1 also contains eight other rows. The method of establishing the elements which occur in these other rows will now be indicated.

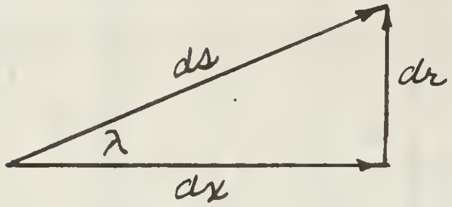
Consider the equation

$$du_x = u_{xx}\, dx + u_{xr}\, dr + u_{xt}\, dt \qquad (19.1)$$

TABLE 19.1 CHARACTERISTIC MATRIX

	1	2	3	4	5	6	7	8	9	10	11	12
	u_{xx}	u_{xr}	u_{rr}	v_{xx}	v_{xr}	v_{rr}	p_{xx}	p_{xr}	p_{rr}	ρ_{xx}	ρ_{xr}	ρ_{rr}
1	$\cos\lambda$	$\sin\lambda$	0	0	0	0	0	0	0	0	0	0
2	$\frac{4}{3}$	0	1	0	$\frac{1}{3}$	0	0	0	0	$\frac{2}{3}(2\kappa+1)\frac{u}{\rho}$	$(\frac{2\kappa+1}{6})\frac{v}{\rho}$	$(\frac{2\kappa+1}{2})\frac{u}{\rho}$
3	0	$\cos\lambda$	$\sin\lambda$	0	0	0	0	0	0	0	0	0
4	0	0	0	$\cos\lambda$	$\sin\lambda$	0	0	0	0	0	0	0
5	0	$\frac{1}{3}$	0	1	0	$\frac{4}{3}$	0	0	0	$(\frac{2\kappa+1}{2})\frac{v}{\rho}$	$(\frac{2\kappa+1}{6})(\frac{u}{\rho})$	$\frac{2}{3}(2\kappa+1)\frac{v}{\rho}$
6	0	0	0	0	$\cos\lambda$	$\sin\lambda$	0	0	0	0	0	0
7	0	0	0	0	0	0	$\cos\lambda$	$\sin\lambda$	0	0	0	0
8	ρu	0	ρu	ρv	0	ρv	$\frac{1}{(\gamma-1)}$	0	$\frac{1}{(\gamma-1)}$	0	0	0
9	0	0	0	0	0	0	0	$\cos\lambda$	$\sin\lambda$	0	0	0
10	0	0	0	0	0	0	0	0	0	$\cos\lambda$	$\sin\lambda$	0
11	0	0	0	0	0	0	0	0	0	1	0	1
12	0	0	0	0	0	0	0	0	0	0	$\cos\lambda$	$\sin\lambda$

Let ds be a small displacement in the plane at angle λ as shown. Thus



$$\begin{aligned} dx &= ds \cos\lambda \\ dr &= ds \sin\lambda \end{aligned} \tag{19.2}$$

Let us analyze the changes along ds at some fixed instant of time, so that we may set

$$dt = 0 \tag{19.3}$$

It then follows from Eqs. (19.1), (19.2) and (19.3) that

$$\left(\frac{du_x}{ds}\right) = u_{xs} = u_{xx} \cos\lambda + u_{xr} \sin\lambda \tag{19.4}$$

Similarly

$$\left(\frac{du_r}{ds}\right) = u_{rs} = u_{xr} \cos\lambda + u_{rr} \sin\lambda \tag{19.5}$$

Eq. (19.4) may be recognized as row 1 of Table 19.1. Eq. (19.5) is row 3. The extension of the above method to the other rows of the table should now be evident.

Table 19.1 now represents a set of 12 simultaneous equations which may be written symbolically as

$$[A]\{x\} = \{y\} \tag{19.6}$$

We introduce the following notation

$D(\lambda)$ = determinant of the matrix [A]

$N_i(\lambda)$ = determinant of the array formed by replacing the ith column of the above array by the column vector $\{y\}$. (19.7)

Then by Cramer's rule

$$x_i = \frac{N_i(\lambda)}{D(\lambda)} \tag{19.8}$$

Notice that both determinants in Eq. (19.8) appear to be function of angle λ because some of the elements in these arrays are functions of λ.

We now ask whether there are any characteristic values of λ such that

$$D(\lambda) = 0 \tag{19.9}$$

If Eq. (19.9) be satisfied for certain values of λ, then since x_i is finite, it follows from Eq. (19.8) that for these same values of λ

$$N_i(\lambda) = 0 \tag{19.10}$$

Consequently, Eq. (19.78 now assumes the indeterminate form

$$x_i = \frac{0}{0} \tag{19.11}$$

Thus, if there exist certain characteristic values of λ for which Eq. (19.9) is satisfied, then these values define the directions of characteristic lines along which the second derivatives denoted by the variables x_i may be indeterminate or discontinuous. Under these circumstances the above system of equations is said to by hyperbolic.

Conversely, if there exist no values of λ for which Eq. (19.9) can be satisfied, then there are no such characteristic lines. Consequently, the second derivatives denoted by the variables x_i are everywhere determinate and continuous. Under these circumstances the above system of equations is said to be elliptical.

Hence the classification of our four basic equations as elliptical or hyperbolic hinges on the value of the determinant of the 12 by 12 array listed in Table 19.1. This determinant has been evaluated in section 21. Inspection of section 21 will reveal that this evaluation was a lengthy and arduous task. Fortunately, the final result obtained proves to be extremely simple. It is

$$D = +\frac{4}{3}\left(\frac{1}{\gamma-1}\right) \neq 0 \qquad (19.12)$$

This result is very interesting. Notice firstly that the elements of the matrix as listed in Table 19.1 are functions not only of λ, but also of parameters γ and κ, and also of the three fundamental variables u, v and ρ. Curiously enough, they are not functions of the fourth fundamental variable p which does not appear in the table.

Under these circumstances one would naturally expect the final determinant itself to be some function not only of λ, but also of $\gamma, \kappa, u, v,$ and ρ. But the result turns out differently. Eq. (19.12) shows that all of the above quantities except γ cancel out of the final determinant!

This result proves that the four basic equations here considered, namely, Eq. (18.22) through (18.25), are in themselves unconditionally elliptic in character, irrespective of local Mach number or anything else!

Of course as we have already seen, when the second law of thermodynamics, Eq. (18.34), is added to the system, the equations change from elliptical to piecewise parabolic.

20. Derivation of Characteristic Matrix

In this section we derive the elements of the characteristic matrix as previously displayed in rows 2, 5, 8 and 11 of Table 19.1. These represent, respectively, the momentum equation in direction x, the momentum equation in direction r, the energy equation and the continuity equation. The elements in the other eight lines of the table, which are either $\cos\lambda$, $\sin\lambda$, or zero, have already been established and explained in section 19.

For the momentum relation in direction x our starting point in Eq. (18.22) as follows

$$\left(\frac{\partial u}{\partial t}\right) = - u \left(\frac{\partial u}{\partial x}\right) - v \left(\frac{\partial u}{\partial r}\right) - \frac{1}{\rho} \left(\frac{\partial P}{\partial x}\right) + f_x \tag{20.1}$$

We now substitute for the various unknowns in terms of the primary variables u, v, p, ρ and of their various derivatives according to Eqs. (18.7) through (18.21). It suffices for our present purpose, however, to show explicitly only the terms in the twelve variables u_{xx}, u_{xr}, u_{rr}, v_{xx}, v_{xr}, v_{rr}, p_{xx}, p_{xr}, p_{rr}, ρ_{xx}, ρ_{xr}, ρ_{rr}. These are the only terms which occur in the main matrix on the left side of the final system of twelve simultaneous equations. All other terms are ultimately transferred to the right side of these equations. Such unessential terms are of no immediate interest and may simply be dropped from the present derivation. These unessential missing terms are indicated in the expressions below by rows of dots.

Thus Eqs. (20.1) now reduces as follows.

$$\left(\frac{\partial u}{\partial t}\right) = f_x + \ldots. \tag{20.2}$$

Substituting from Eq. (18.19) and simplifying

$$\begin{aligned}\left(\frac{\partial u}{\partial t}\right) &= \frac{1}{\rho} \left\{\left(\frac{\partial \tau_{xx}}{\partial x}\right) + \frac{1}{r}\frac{\partial}{\partial r} \left(r\tau_{rr}\right)\right\} + \ldots. \\ &= \frac{1}{\rho} \left\{\left(\frac{\partial \tau_{xx}}{\partial x}\right) + \left(\frac{\partial \tau_{rr}}{\partial r}\right) + \ldots.\right\} + \ldots.\end{aligned} \tag{20.3}$$

Substituting from Eqs. (18.14) and (18.17) and simplifying further

$$\begin{aligned}\left(\frac{\partial u}{\partial t}\right) = \frac{\varepsilon}{\rho} \Bigg\{ &\frac{\partial}{\partial x} \left\{\frac{2\rho}{3} \left[2 \left(\frac{\partial u}{\partial x}\right) - \left(\frac{\partial v}{\partial r} + \ldots.\right)\right] + \left(\frac{2\kappa+1}{3}\right)\left[2u \left(\frac{\partial \rho}{\partial x}\right) - v \left(\frac{\partial \rho}{\partial r}\right)\right]\right\} \\ &+ \frac{\partial}{\partial r} \left\{\rho \left[\left(\frac{\partial v}{\partial x}\right) + \left(\frac{\partial u}{\partial r}\right)\right] + \left(\frac{2\kappa+1}{2}\right)\left[v \left(\frac{\partial \rho}{\partial x}\right) + u \left(\frac{\partial \rho}{\partial x}\right)\right]\right\}\Bigg\} + \ldots.\end{aligned}$$

$$= \frac{\varepsilon}{\rho} \left\{ \frac{2\rho}{3} [2u_{xx} - v_{xr}] + (\frac{2\kappa+1}{3}) [2u\rho_{xx} - v\rho_{xr}] \right.$$

$$\left. + \rho [v_{xr} + u_{rr}] + (\frac{2\kappa+1}{2}) [v\rho_{xr} + u\rho_{rr}] \right\} + \ldots . \tag{20.4}$$

Upon regrouping terms we finally obtain

$$(\frac{\partial u}{\partial t}) = \varepsilon \left\{ \frac{4}{3} u_{xx} + u_{rr} + \frac{1}{3} v_{xr} + \frac{2}{3} (2\kappa+1) \frac{u}{\rho} \rho_{xx} \right.$$

$$\left. + (\frac{2\kappa+1}{6}) \frac{v}{\rho} \rho_{xr} + (\frac{2\kappa+1}{2}) \frac{u}{\rho} \rho_{rr} \right\} + \ldots . \tag{20.5}$$

This result now accounts for the elements in row 2 of Table 19.1. Notice that the common factor ε is omitted from the coefficients in the table.

A similar procedure applies to the momentum relation in the r direction, Eq. (18.23). Thus

$$(\frac{\partial v}{\partial t}) = - u (\frac{\partial v}{\partial x}) - v (\frac{\partial v}{\partial r}) - \frac{1}{\rho} (\frac{\partial P}{\partial r}) + f_r$$

$$= f_r + \ldots . = \frac{1}{\rho} \left\{ (\frac{\partial \tau_{xr}}{\partial x}) + \frac{1}{r} \frac{\partial}{\partial r} (r\tau_{rr}) - \frac{\tau_{\theta\theta}}{r} \right\} + \ldots .$$

$$= \frac{1}{\rho} \left\{ (\frac{\partial \tau_{xr}}{\partial x}) + (\frac{\partial \tau_{rr}}{\partial r}) \right\} + \ldots .$$

$$= \frac{\varepsilon}{\rho} \left\{ \frac{\partial}{\partial x} \left\{ \rho \left[(\frac{\partial v}{\partial x}) + (\frac{\partial u}{\partial r}) \right] + (\frac{2\kappa+1}{2}) \left[v (\frac{\partial \rho}{\partial x}) + u (\frac{\partial \rho}{\partial r}) \right] \right\} \right.$$

$$\left. + \frac{\partial}{\partial r} \left\{ \frac{2\rho}{3} \left[2 (\frac{\partial v}{\partial r}) - (\frac{\partial u}{\partial x}) + \ldots . \right] + (\frac{2\kappa+1}{3}) \left[- u (\frac{\partial \rho}{\partial x}) + 2v (\frac{\partial \rho}{\partial r}) \right] \right\} \right\} + \ldots .$$

$$= \varepsilon \left\{ [v_{xx} + u_{xr}] + (\frac{2\kappa+1}{2}) \left[\frac{v}{\rho} \rho_{xx} + \frac{u}{\rho} \rho_{xr} \right] \right.$$

$$\left. + \frac{2}{3} [2v_{rr} - u_{xr}] + \frac{2}{3} (2\kappa+1) [- u\rho_{xr} + 2v\rho_{rr}] \right\} + \ldots . \tag{20.6}$$

Finally

$$(\frac{\partial v}{\partial t}) = \varepsilon \left\{ \frac{1}{3} u_{xx} + v_{xx} + \frac{4}{3} v_{rr} + (\frac{2\kappa+1}{2}) \frac{v}{\rho} \rho_{xx} \right.$$

$$\left. + (\frac{2\kappa+1}{6}) \frac{u}{\rho} \rho_{xr} + \frac{2}{3} (2\kappa+1) \frac{v}{\rho} \rho_{rr} \right\} + \ldots . \tag{20.7}$$

This result now accounts for the elements in row 5 of Table 19.1.

It is most convenient to consider next the continuity relation, Eq. (18.24). Thus

$$(\frac{\partial \rho}{\partial t}) = - \frac{\partial}{\partial x}\left[\rho u - \kappa\varepsilon\, (\frac{\partial \rho}{\partial x})\right] - \frac{1}{r}\frac{\partial}{\partial r}\left[r(\rho v - \kappa\varepsilon\, \frac{\partial \rho}{\partial r})\right]$$

$$= + \kappa\varepsilon\,\{\rho_{xx} + \rho_{rr}\} + \ldots . \tag{20.8}$$

This result accounts for the elements in row 11 of Table 19.1.

Finally we consider the energy relation, Eq. (18.25). It is convenient to write this as

$$\frac{1}{(\gamma-1)}\,(\frac{\partial p}{\partial t}) = \dot{Q} - (\frac{u^2 + v^2}{2} + E)\,(\frac{\partial \rho}{\partial t}) - \rho\left[u\,(\frac{\partial u}{\partial t}) + v\,(\frac{\partial u}{\partial t}) + (\frac{\partial E}{\partial t})\right] \tag{20.9}$$

Substituting for $\dot{Q}$ from Eq. (18.21), then expanding and dropping unessential terms gives

$$\frac{1}{(\gamma-1)}\,(\frac{\partial p}{\partial t}) = - \frac{\partial}{\partial x}\left[Hu - \kappa\varepsilon\, (\frac{\partial Q}{\partial x})\right] - \frac{1}{r}\frac{\partial}{\partial r}\left[r\,(Hv - \kappa\varepsilon\, \frac{\partial Q}{\partial r})\right]$$

$$+ \dot{W} + \rho f_x u + \rho f_r v$$

$$- (\frac{u^2 + v^2}{2} + E)\,(\frac{\partial \rho}{\partial t}) - \rho\left[u\,(\frac{\partial u}{\partial t}) + v\,(\frac{\partial v}{\partial t}) + \ldots .\right]$$

$$= \kappa\varepsilon\left[(\frac{\partial^2 Q}{\partial x^2}) + (\frac{\partial^2 Q}{\partial r^2})\right] - (\frac{u^2 + v^2}{2} + E)\,(\frac{\partial \rho}{\partial t})$$

$$- \rho u\,(\frac{\partial u}{\partial t} - f_x) - \rho v\,(\frac{\partial v}{\partial t} - f_r) + \ldots . \tag{20.10}$$

Inspection of Eq. (20.2) and of the early development of Eq. (20.6) reveals that all second derivatives cancel out of the quantities $(\frac{\partial u}{\partial t} - f_x)$ and $(\frac{\partial v}{\partial t} - f_r)$. These terms therefore vanish from Eq. (20.10).

By evaluating the derivatives $(\frac{\partial^2 Q}{\partial x^2})$ and $(\frac{\partial^2 Q}{\partial r^2})$ in Eq. (20.10) from Eq. (18.8) , and substituting for $(\frac{\partial \rho}{\partial t})$ from Eq. (20.8), we obtain

$$\frac{1}{(\gamma-1)}\left(\frac{\partial p}{\partial t}\right) = \kappa\varepsilon \left\{ \frac{1}{(\gamma-1)} (p_{xx} + p_{rr}) + \rho u (u_{xx} + u_{rr}) + \rho v (v_{xx} + v_{rr}) \right.$$

$$\left. + \left(\frac{u^2 + v^2}{2} + E\right) (\rho_{xx} + \rho_{rr}) + \dots \right\}$$

$$- \left(\frac{u^2 + v^2}{2} + E\right) (\kappa\varepsilon) (\rho_{xx} + \rho_{rr}) + \dots \qquad (20.11)$$

Notice that the terms in $(\rho_{xx} + \rho_{rr})$ cancel out of Eq. (20.11). Hence multiplying through by $(\gamma-1)$ and rearranging, we finally obtain

$$\left(\frac{\partial p}{\partial t}\right) = \kappa\varepsilon\,(\gamma-1) \left\{ \rho u (u_{xx} + u_{rr}) + \rho v (v_{xx} + v_{rr}) + \frac{(p_{xx} + p_{rr})}{(\gamma-1)} \right\} + \dots \qquad (20.12)$$

This result accounts for the elements in row 8 of Table 19.1.

Thus all elements of the characteristic matrix summarized in Table 19.1 have now been accounted for in complete detail.

21. Evaluation of Characteristic Determinant

The discussion in section 8 shows that it is not practical to evaluate the 12 by 12 determinant of Table 19.1 by the general method of successive expansions which is so convenient when dealing with a small determinant, say of size 4 by 4, for example. Instead we choose here to transform our original matrix into upper triangular form. We say a matrix is in upper triangular form when all elements below the main diagonal are zeros. It can easily be shown that the determinant of a matrix in upper triangular form is just equal to the product of the elements which lie along the main diagonal.

In reducing an arbitrary square matrix to upper triangular form, we make use of the following two theorems:

1) If the elements of any row are multiplied by an arbitrary constant, the determinant is multiplied by the same constant.

2) If to the elements of any row are added an arbitrary multiple of the respective elements of any other row, the determinant remains unchanged.

Consider a general transformation in which the elements a_{mk} of row m are replaced by new values a'_{mk} as defined below. The second index k denotes the column number; it should be understood to range over all of the columns. Let a_{nk} be the elements of some other row n, and let c_1 and c_2 be two constants which remain to be defined. The general transformation of interest may now be expressed as follows.

$$a'_{mk} = c_1\, a_{mk} + c_2\, a_{nk} \tag{21.1}$$

Let us now choose these two constants in the following way

$$c_1 = a_{n\ell} \qquad c_2 = -\, a_{m\ell} \tag{21.2}$$

where index ℓ denotes some particular fixed column. It now follows that in general

$$a'_{mk} = a_{n\ell}\, a_{mk} - a_{m\ell}\, a_{nk} \tag{21.3}$$

but that for the particular column ℓ

$$a'_{m\ell} = a_{n\ell}\, a_{m\ell} - a_{m\ell}\, a_{n\ell} \equiv 0 \tag{21.4}$$

The above transformation may therefore be used to reduce the value of the element in row m, column ℓ, to zero. Repeated and systematic use of this technique may therefore be employed to reduce an arbitrary matrix to upper triangular form.

It now follows from the above theorems that this transformation changes the determinant by the known factor

$$c_1 = a_{n\ell} \tag{21.5}$$

Hence we can keep track of the changes in the determinant as these successive transformations are carried out.

We now summarize below the sixteen successive transformations that were actually used to reduce the original matrix of Table 19.1 to upper triangular format.

$$a'_{2k} = (\cos\lambda)\ a_{2k} - \frac{4}{3}\ a_{1k} \tag{21.6}$$

$$a'_{8k} = (\cos\lambda)\ a_{8k} - \rho u\ a_{1k} \tag{21.7}$$

$$a'_{3k} = \left(\frac{4}{3}\ \sin\lambda\right)\ a_{3k} + (\cos\lambda)\ a_{2k} \tag{21.8}$$

$$a'_{5k} = (4\ \sin\lambda)\ a_{5k} + a_{2k} \quad = \tag{21.9}$$

$$a'_{8k} = (\cos\lambda)\ a_{8k} + (\rho u\ \sin\lambda)\ a_{3k} \tag{21.10}$$

$$a'_{5k} = \left(1 + \frac{\sin^2\lambda}{3}\right)\ a_{5k} - (\cos\lambda)\ a_{3k} \tag{21.11}$$

$$a'_{8k} = \left(1 + \frac{\sin^2\lambda}{3}\right)\ a_{8k} - \rho u\ a_{3k} \tag{21.12}$$

$$a'_{5k} = (\cos\lambda)\ a_{5k} - 4\ \sin\lambda\ \left(1 + \frac{\sin^2\lambda}{3}\right)\ a_{4k} \tag{21.13}$$

$$a'_{8k} = (\cos\lambda)\ a_{8k} - \rho v\ \cos^2\lambda\ \left(1 + \frac{\sin^2\lambda}{3}\right)\ a_{4k} \tag{21.14}$$

$$a'_{6k} = \left[\frac{32}{9} \sin^2\lambda \left(1 + \frac{\sin^2\lambda}{2}\right)\right] a_{6k} + (\cos\lambda)\, a_{5k}$$

$$a'_{8k} = (\cos\lambda)\, a_{8k} + \rho \cos^2\lambda \left[\frac{u \cos\lambda}{3} + v \sin\lambda \left(1 + \frac{\sin^2\lambda}{3}\right)\right] a_{6k} \qquad (21.16)$$

$$a'_{8k} = \left(\frac{16}{3} \sin\lambda\right) a_{8k} - \rho \cos^2\lambda \left[\frac{u \sin\lambda \cos\lambda}{3} + v \left(1 + \frac{\sin^2\lambda}{3}\right)\right] a_{6k} \qquad (21.17)$$

$$a'_{8k} = (\cos\lambda)\, a_{8k} - \frac{16}{3} \frac{\sin\lambda \cos^4\lambda}{(\gamma-1)} \left(1 + \frac{\sin^2\lambda}{3}\right) a_{7k}$$

$$a'_{9k} = \frac{16}{3} \frac{\sin^2\lambda \cos^4\lambda}{(\gamma-1)} \left(1 + \frac{\sin^2\lambda}{3}\right) a_{9k} + (\cos\lambda)\, a_{8k}$$

$$a'_{11k} = (\cos\lambda)\, a_{11k} - a_{10k} \qquad (21.20)$$

$$a'_{12k} = (\sin\lambda)\, a_{12k} + (\cos\lambda)\, a_{11k} \qquad (21.21)$$

The sixteen successive transformations defined by Eqs. (21.6) through (21.21) above involve intermediate calculation steps which are too voluminous to present in detail here. Since our purpose is simply to evaluate the determinant, the only details of the final upper triangular matrix of interest here are the values of the twelve elements along the main diagonal. These are summarized below.

$$a_{11} = \cos\lambda \qquad a_{77} = \cos\lambda$$

$$a_{22} = -\frac{4}{3} \sin\lambda \qquad a_{88} = -\frac{16}{3} \frac{\sin^2\lambda \cos^4\lambda}{(\gamma-1)} \left(1 + \frac{\sin^2\lambda}{3}\right)$$

$$a_{33} = \left(1 + \frac{\sin^2\lambda}{3}\right) \qquad a_{99} = \frac{16}{3} \frac{\sin\lambda \cos^4\lambda}{(\gamma-1)} \left(1 + \frac{\sin^2\lambda}{3}\right) \qquad (21.22)$$

$$a_{44} = \cos\lambda \qquad a_{10,10} = \cos\lambda$$

$$a_{55} = -\frac{32}{9} \sin^2\lambda \left(1 + \frac{\sin^2\lambda}{2}\right) \qquad a_{11,11} = -\sin\lambda$$

$$a_{66} = \frac{16}{3} \sin\lambda \qquad a_{12,12} = +1$$

The value D' of the determinant of the matrix in upper diagonal form is just the product of the twelve factors given in Eqs. (21.22). This is

$$D' = \frac{8}{9} \left(\frac{16}{3}\right)^4 \frac{\cos^{12}\lambda \sin^8\lambda}{(\gamma-1)^2} \left(1 + \frac{\sin^2\lambda}{2}\right) \left(1 + \frac{\sin^2\lambda}{3}\right)^3 \tag{21.23}$$

The changes in the value D of the original determinant produced by the sixteen transformations listed in Eqs. (21.6) through (21.21) is equal to the product of the sixteen factors in these equations. In each case the relevant factor is the coefficient of the first term on the right side of the equation. The product of these factors is

$$C = \frac{2}{3} \left(\frac{16}{3}\right)^4 \frac{\cos^{12}\lambda \sin^8\lambda}{(\gamma-1)} \left(1 + \frac{\sin^2\lambda}{2}\right) \left(1 + \frac{\sin^2\lambda}{3}\right)^3 \tag{21.24}$$

Hence the value D of the determinant of the original matrix shown in Table 19.1 is finally

$$\frac{D'}{C} = D = \frac{4}{3} \frac{1}{(\gamma-1)} \neq 0 \tag{21.25}$$

This completes the evaluation of the characteristic determinant.

22. Symbols

a	Spacing of finite difference mesh
a_{mk}	Matrix coefficient, row m, column k
a'_{mk}	Transformed matrix coefficient, row m, column k
A,B,C,D	Coefficients in governing equation for nonturbulent compressible flow
A,B	Constants in logarithmic law of the wall
c	Sonic velocity
c_1, c_2	Constants in matrix transformation equation
C	Product of changes in determinant resulting from matrix transformations
D	Characteristic determinant of original matrix
D'	Characteristic determinant of matrix transformed to upper triangular form
D	Rate of dissipation of turbulent energy
e	Internal energy per unit mass
$\vec{e}_x$, $\vec{e}_r$, $\vec{e}_\theta$	Unit vectors in cylindrical coordinates
E	Mean kinetic energy of turbulent fluctuations per unit mass
$\vec{f}$	Net force per unit mass exerted on fluid element by deviatoric Reynolds stresses
f_x, f_r	Components of $\vec{f}$
G	Auxiliary variable defined by Eq. (13.22)
H	Total effective enthalpy per unit volume
i,j,k,ℓ,m,n	Indices
I, I_o, I'	Total weighted error with respect to the second law of thermodynamics
L_{cr}	Length from exit plane of nozzle to point where jet attaches to pipe wall
M	Local Mach number
M_x, M_r	Components of M

n	Number of radial stations in finite difference mesh
$\vec{n}$	Outward unit vector normal to contour
N	Total number of internal mesh points in cell. Also total number of adjustable parameters across exit section of cell
N_i	Determinant in Cramer's rule
p	Fluid pressure
P	Total effective pressure of fluid and of turbulent fluctuations
Q	Total energy per unit volume
r	Radial coordinate
R	Perfect gas constant
s	Entropy per unit mass
S	Entropy per unit volume
ds, dx, dr	Elements of infinitesimal triangle
t	Time
T	Absolute temperature
u_i	Velocity component in cartesian tensor notation
u, v, w	Velocity componentts in cylindrical coordinates
u^*	Friction velocity in logarithmic law of the wall
$\vec{V}$	Velocity vector
$\dot{W}$	Time rate of work done by deviatoric Reynolds stresses upon deviatoric strain rates of mean flow
x_i	Coordinate in cartesian tensor notation
x, r, θ	Cylindrical coordinates
z	Error at a point with respect to second law of thermodynamics
α_j	Generalized variable at point along exit cross-section of cell
$\delta\alpha_j$	Small change in α_j

δ_{ij}	The Kronecker delta
ε	Eddy viscosity
κ	Inverse of turbulent Prandtl number
λ	Angle of characteristic with respect to x axis
ϕ	Velocity potential
ψ	Stream function
ρ	Density
τ_{ij}	Deviatoric Reynolds stresses in cartesian tensor notation
τ_{xx}, τ_{xr}, etc.	Deviatoric Reynolds stresses in cylindrical coordinates

<u>Subscript notation for derivatives</u>

$$\frac{\partial\phi}{\partial x} = \phi_x$$

$$\frac{\partial^2\phi}{\partial x\partial r} \quad \phi_{xr}$$

etc.

<u>Notation for fluctuating quantities</u>

$\overline{u} = u$	Ensemble average value of u
u'	Fluctuation of u from ensemble average
$u'' = (u+u')$	Instantaneous value of u

etc.

23. References and Bibliography

No attempt is made here to survey the huge volume of literature in this field, nor does the present report follow closely the approach of any one of the papers listed below. Nevertheless this list is typical of contemporary work in the field and offers useful background information. The present author has found it convenient to group the references very loosely into the seven categories shown, and to list the papers alphabetically by principal author within each category.

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